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Pentagon identity solution in algebra $U_q(sl_N)$

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Work done by
M02-921b student ________________ Alekseev Victor Aleksandrovich

Supervisor:
PhD, Senior Researcher ________________ Sleptsov Alexey Vasilevich

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Abstract

The connection between $U_q(sl_N)$ 6-j symbols and orthogonal polynomials is established. We demonstrate that the pentagon identity alone implies that multiplicity-free 6-j symbols satisfy three-term recurrence relation and orthogonality, making them orthogonal polynomials of several variables, associated with Young diagrams. The coefficients in the recurrence relation are primitive 6-j symbols.

We examine in more detail the case of orthogonal polynomials of one variable, especially when two representations are symmetric and one is conjugate to symmetric. The coefficients can be calculated explicitly via Racah back-coupling relation alone. The 6-j symbol is identified with q-Racah polynomial, that is, a terminating q-hypergeometric series ${}_4\Phi_3$. It includes as a special case the expression conjectured previously for two symmetric and two conjugate to symmetric representations [1]. Thus, the algebraic expression for the considered class of 6-j symbols is obtained from the general setting with the only knowledge about the pentagon identity and the Racah back-coupling rule.

Contents

1 Introduction **3**
1.1 Tensor products of representations in $U_q(sl_N)$ 4
1.2 General properties of multiplicity-free 6-j symbols 7
1.3 q-Hypergeometric series and Racah polynomial 11
1.4 Relation between 6-j symbols and orthogonal polynomials 15

2 Three-term recurrence relation for 6-j symbols **17**
2.1 Three-term relation for 6-j symbols with rectangular representation 17
2.2 Three-term relation for general multiplicity-free 6-j symbols 21

3 Pentagon solution for types I^+ and II^+ **25**
3.1 Orthogonal polynomials for type I^+ 25
3.2 Orthogonal polynomials for type II^+ 30

A Racah matrices from back-coupling rule **35**
A.1 Back-coupling rule solution 35
A.2 Solution uniqueness 36
A.3 Two-dimensional Racah matrices 38

1 Introduction

The representation theory is widely used to study symmetries and classify them with analytical methods. The symmetry groups arise in a lot of areas of classical and quantum physics. The questions about representations structure can be roughly divided into two parts. The first one is the representation theory of one separate irreducible representation. The second one is the theory of products of representations. This separation is very evident in the Hopf algebras, where inside the representation the structure is given by the algebra multiplication rules, whereas the coalgebra comultiplication governs the connection between the products of different representations. The study of the latter one is known as Racah-Wigner algebra [2].

For quantum algebras there are several important concepts that are central in the description of representation products. The first one is $\mathcal{R}$-matrix and the second one is 3-j symbol (coupling coefficient, Clebsch-Gordan coefficient) and the third one is 6-j symbol (recoupling coefficient). The 3-j symbol is probably the most celebrated object among these three for physics. It allows to describe of the decomposition of the product of two representations into the third one. However, it acts non-trivially in the representation space and thus depends on the basis choice. The next two objects acts on each irreducible representation as a constant, so they are invariant objects of the representation tensor product. $\mathcal{R}$-matrix has a meaning of representations permutation and acts non-trivially in quantum algebra with $q \neq 1$. The 6-j symbol arises as the associator in the representations product.

There is a closed expression for $\mathcal{R}$-matrix [3], but 3-j symbols and 6-j symbols are lacking of such description. Only for the simplest case of quantum Lie algebra $U_q(sl_2)$ there is an expression for both of them [4]. There is an algorithm how to calculate them by the definition, but it soon becomes a cumbersome task. Although they can be obtained explicitly, we are interested in the analytical closed expression for a general $U_q(sl_N)$ symbol.

For the 6-j symbols beyond $U_q(sl_2)$ only a few series are known. Firstly, there is an expression for 6-j symbol with three symmetric representations [5]. Secondly, there are expressions for Racah matrices that have only symmetric or conjugate to symmetric representations [1, 6]. Also there is an formula for one symmetric and one antisymmetric representations. All of them are conjectural in a sense that they are not proven, but checked to be true for all known cases.

This paper provides a new framework that can be used to obtain 6-j symbols analytically. It consider 6-j symbols as abstract functions that satisfy several identities. Namely, 6-j symbols satisfy pentagon identity, Yang-Baxter equation, orthogonality and possess tetrahedral symmetries [7]. The central idea of this paper is that this information is enough to uniquely determine the value of 6-j symbols. It turns out that these properties can be translated into the language of orthogonal polynomials. All orthogonal polynomials possess a three-term recurrence relation. The pentagon identity implies the three-term recurrence relation on certain $U_q(sl_N)$ 6-j symbols. This recurrence can be solved explicitly, which provide an algebraic expression for 6-j symbols fully determined by its properties.

For instance, $U_q(sl_2)$ 6-j symbols are known to be a properly normalized q-Racah polynomial. The orthogonality of such polynomial coincides with 6-j symbol orthogonality. A similar method was used in the work [2] to derive arbitrary $U_q(su_2)$ 6-j symbol. For $U_q(sl_N)$ the situation becomes much more complex, as the pentagon equation interlaces different representations in a far more unpredictable way than for $U_q(sl_2)$. However, we have discovered a way how to transform the pentagon identity in $U_q(sl_N)$ into a three-term recurrence relation on 6-j symbols.

We need to mention that pentagon itself does not provide us with a concrete expression for 6-j symbols. The three-term recurrence relation coefficients are written purely in primitive 6-j symbols, that is, 6-j symbols containing a fundamental representation in arguments at least once. Such symbols itselves are hard to obtain in general. We used Racah back-coupling rule to determine their values for a less general setting deduce the polynomial expression for 6-j symbols with two symmetric representations and one conjugate to symmetric.

The paper is organized as follows.

In Section 1 we introduce the notion of $U_q(sl_N)$ representations, their tensor products. On the space of representation tensor product we define 3-j symbols, 6-j symbols and $\mathcal{R}$-matrices. We emphasize our attention to $\mathcal{R}$-matrix and 6-j symbol properties. Then we recall the notion of basic hypergeometric series, specifically ${}_4\Phi_3$ and the facts from the theory of orthogonal polynomials. Then we review the interplay between 6-j symbols and orthogonal polynomials. Lastly, we state two main results of our paper: the deduction of three-term recurrence relation for a wide class of multiplicity-free 6-j symbols and the derivation of the closed expression for 6-j symbols of types I^+ and II^+.

In Section 2 we prove that for the general multiplicity-free 6-j symbol, that is, with two symmetric representations, it is possible to construct a three-term recurrence relation which corresponds to a multivariable orthogonal polynomial. We consider the polynomial of one variable as a basic example of this statement and write down it in detail. Corresponding 6-j symbols have one rectangular representation and two representations either symmetric, or antisymmetric, or conjugate to symmetric. In this section we only use pentagon identity and basic fusion rules of $U_q(sl_N)$. The resulting three-term relation is written in terms of primitive 6-j symbols. The existence of such relation implies that considered 6-j symbols are orthogonal polynomials, which can be easily obtained recursively.

In Section 3 we apply this method for 6-j symbols of types I^+ and II^+. In this case we are able to explicitly calculate all arising coefficients, using the results from Appendix A, where we solved Racah back-coupling rule for two-dimensional case. After some simplifications we write both types of 6-j symbols as a q-Racah polynomial. This polynomial has an explicit form in terms of terminating q-hypergeometric series ${}_4\Phi_3$. Thus, we obtain an expression for 6-j symbols from the pentagon identity and Racah back-coupling rule.

In Appendix A we solve the back-coupling rule with respect to 6-j symbols, using the spectral decomposition of $\mathcal{R}$-matrices. This result has already been known for some time in papers on knot polynomials [8]. However, we include it in our work, because it is essential in the derivation. This expression allow to evaluate any Racah matrix of dimension 2, which we use extensively in the main part of the paper. We also add the criteria for the solution uniqueness and prove that at least for the considered in this work primitive 6-j symbols the expression is unique.

1.1 Tensor products of representations in $U_q(sl_N)$

Irreducible repesentations of algebra $U_q(sl_N)$. We will consider $U_q(sl_N)$ algebra with finite-dimensional representations of the highest weight for the rest of the paper. More precisely, we only use the notion of Young diagrams and fusion rules for them [9].

Irreducible representation R_μ of $U_q(sl_N)$ is determined by its highest weight. All of them can be described by the ordered set of non-increasing non-negative integers $\mu_1 \geq \mu_2 \geq \cdots \geq \mu_N$, it is called a Young diagram. We denote it as $\mu = [\![\mu_1, \ldots, \mu_N]\!]$. For $U_q(sl_N)$ irreducible representations μ and $\mu' = [\![\mu_1 + a, \mu_2 + a, \ldots, \mu_N + a]\!]$ are equivalent, thus, we can without loss of generality set $\mu_N = 0$. The order of Young diagram $|\mu|$ is just the number of elements in the partition: $|\mu| = \sum_i \mu_i$.

If for $k < N$ the diagram has $\mu_{k+1} = 0$ we write it just as $[\![\mu_1, \ldots \mu_k]\!]$, in particular, symmetric representations are denoted as $[\![\mu_1]\!]$. A representation $R_\nu = \overline{R_\mu}$ is conjugate to R_μ, it has Young diagram $\nu = [\![\mu_1 - \mu_N, \mu_1 - \mu_{N-1}, \ldots, \mu_1 - \mu_2, 0]\!]$. For the diagram with n equal rows we write μ_i^n instead of $\mu_i, \mu_i, \ldots, \mu_i$, thus, conjugate to symmetric representation has diagram $\overline{[\![\mu_1]\!]} = [\![\mu_1^{N-1}]\!]$.

There is a natural q-analogue for trace and dimension [10, 5] in the finite-dimensional quantum algebras defined in terms of Cartan subalgebra H and weights α_i, $1 \le i \le N-1$:

$$\begin{aligned} &\mathrm{Tr}_q(z) := \mathrm{Tr}\left(K_{2\rho} z\right), \qquad K_{2\rho} = q^{\left(\sum_i \alpha_i, H\right)}, \\ &\dim_q V := \mathrm{Tr}_q(id_V) \end{aligned} \tag{1}$$

The quantum dimension of an irreducible $U_q(sl_N)$ representation can be written explicitly [5]:

$$\dim_q V_\mu \equiv D_\mu = \prod_{(i,j)\in\lambda} \frac{[N+i-j]}{[\mu_i - i + \mu^T_j - j + 1]}, \qquad [x] := \frac{q^{x/2} - q^{-x/2}}{q^{1/2} - q^{-1/2}} \tag{2}$$

where μ^T is a transposed Young diagram and $[x]$ is a quantum number. The two types of brackets are introduced to avoid confusion. Quantum dimension and trace becomes the ordinary dimension and trace when $q \to 1$.

Tensor product of irreducible representations. Let us consider the tensor product of two irreducible representations $R_\mu \otimes R_\nu$ acting in the space $V_\mu \otimes V_\nu$ in a obvious way. This product is itself a representation, but in general reducible, hence it can be decomposed into irreducible ones by Littlewood-Richardson rules [9]. On the level of vector spaces the decomposition is

$$V_\mu \otimes V_\nu = \bigoplus_\rho M^\rho_{\mu\nu} \otimes V_\rho \tag{3}$$

Here $M^\rho_{\mu\nu}$ is the multiplicity space, i.e. the vector space of highest weight ρ in the product. The dimension $m = \dim(M^\rho_{\mu\nu})$ is equal to the number of V_ρ in the decomposition. If $M^\rho_{\mu\nu}$ is one-dimensional, the representation is called multiplicity-free and $M^\rho_{\mu\nu} \otimes V_\rho$ is canonically identified with V_ρ. In general $m \neq 1$ and it is called the multiplicity of the product. The basis in $M^\rho_{\mu\nu}$ can not be fixed by means of representation theory and should be chosen by other methods.

In each space V_μ there is an eigenbasis given by the highest weight vector and its descendants. With this basis in each irreducible representation and multiplicity space basis specification, one may find an invertible map corresponding to decomposition (3). If we additionally introduce a norm on the space it can be chosen to be unitary. This map as a finite-dimensional matrix acts non-trivially in both representation and multiplicity indices. The components of such matrix are known as Clebsch-Gordan coefficients.

6j-symbols. The tensor product of three representations has the structure of highest weights and it depends on the product order, i.e. it is non-associative. The vector spaces tensor product $(V_1 \otimes V_2) \otimes V_3$ and $V_1 \otimes (V_2 \otimes V_3)$ are canonically isomorphic. This isomorphism can be lifted to the representations, so we have

$$(R_1 \otimes R_2) \otimes R_3 \cong R_1 \otimes (R_2 \otimes R_3) \tag{4}$$

This isomorphism is carried out by invertible transformation, called Racah matrix. If the Hopf algebra has a compatible norm, the transformation can be made unitary. If, additionally, the algebra is invariant under complex conjugation, then the Racah matrix becomes orthogonal. We mostly consider the latter case, which enforces either $q^2 \in \mathbb{R}$ or $|q| = 1$ [3].

A tensor product of three arbitrary irreducible representations R_1, R_2, R_3 acts in the space $V_1 \otimes V_2 \otimes V_3$. It can be decomposed into the direct sum of irreducible representations with possible multiplicities. Introducing representations $X_i \subset R_1 \otimes R_2, Y_j \subset R_2 \otimes R_3, R_\mu \subset R_1 \otimes R_2 \otimes R_3$, we can write:

$$\begin{aligned}(R_1 \otimes R_2) \otimes R_3 &= \left(\bigoplus_i M_{X_i}^{R_1,R_2} \otimes X_i\right) \otimes R_3 = \bigoplus_{i,\mu} M_{X_i}^{R_1,R_2} \otimes M_{R_\mu}^{X_i,R_3} \otimes R_\mu, \\ R_1 \otimes (R_2 \otimes R_3) &= R_1 \otimes \left(\bigoplus_j M_{Y_j}^{R_2,R_3} \otimes Y_j\right) = \bigoplus_{j,\mu} M_{R_\mu}^{R_1,Y_j} \otimes M_{Y_j}^{R_2,R_3} \otimes R_\mu.\end{aligned} \tag{5}$$

The representations in both decompositions are isomorphic, as it can be shown from character theory. However, the isomorphism between multiplicity spaces of the same R_μ is not trivial. As Racah matrix acts non-trivially only in the multiplicity space, the isomorphism can be written as follows.

Definition 1. Racah coefficients are elements of Racah matrix that is the map:

$$U\begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix} : \bigoplus_i M_{X_i}^{R_1,R_2} \otimes M_{R_4}^{X_i,R_3} \to \bigoplus_j M_{R_4}^{R_1,Y_j} \otimes M_{Y_j}^{R_2,R_3}. \tag{6}$$

Definition 2. Wigner 6-j symbol is the element of a normalized Racah matrix:

$$\begin{Bmatrix} R_1 & R_2 & X_i \\ R_3 & R_4 & Y_j \end{Bmatrix} = \frac{1}{\sqrt{D_{X_i} D_{Y_j}}} U_{i,j} \begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix}. \tag{7}$$

If multiplicity occurs, X_i may coincide and thus should have additional multiplicity indices. In this paper we work with multiplicity-free case, so the arguments are in one to one correspondence with Young diagrams. Thus, we do not distinguish representations and corresponding Young diagrams inside the arguments of 6-j symbols.

The Racah matrix can be explicitly written in terms of Clebsch-Gordan coefficients as the composition of four unitary maps, acting in the space $V_1 \otimes V_2 \otimes V_3$. However, the Racah matrix has a trivial indices inside every irreducible representation, this means it is an invariant tensor and can be written canonically.

$\mathcal{R}$-matrix. Algebra $U_q(sl_N)$ has a quasitriangular Hopf algebra structure [3]. It means there is an invertible algebra element $\mathcal{R}$ called $\mathcal{R}$-matrix. We do not define the $\mathcal{R}$-matrix from Hopf algebra and refer to [3] for details.

Let us first fix the notation. The representation of $\mathcal{R}$-matrix is denoted by $\mathcal{R}_{i,j}$ and acts on $V_i \otimes V_j$. We introduce the permutation operator $P_{i,j} : V_i \otimes V_j \to V_j \otimes V_i$ given by

$$P_{1,2}(v_1 \otimes v_2) = v_2 \otimes v_1 \qquad \forall v_1 \in V_1, \forall v_2 \in V_2 \tag{8}$$

We can compare representations $R_1 \otimes R_2$ and $P_{1,2}(R_2 \otimes R_1)P_{1,2}^{-1}$. In $U(sl_N)$ these representations are equivalent, but in $U_q(sl_N)$ for $q \neq 1$ they are different due to the non-cocommutativity of the Hopf algebra. Operator $\hat{\mathcal{R}}_{1,2} = P_{1,2}\mathcal{R}_{1,2}$ restores the equivalence between them. For any representations R_1, R_2:

$$\hat{\mathcal{R}}_{1,2}(R_1 \otimes R_2)\hat{\mathcal{R}}_{1,2}^{-1} = R_2 \otimes R_1, \tag{9}$$

Also $\mathcal{R}$-matrix have to satisfy so-called hexagon axioms:

$$\begin{aligned}\hat{\mathcal{R}}_{12,3} &= (\hat{\mathcal{R}}_{1,3} \otimes id)(id \otimes \hat{\mathcal{R}}_{2,3}) \\ \hat{\mathcal{R}}_{1,23} &= (id \otimes \hat{\mathcal{R}}_{1,3})(\hat{\mathcal{R}}_{1,2} \otimes id)\end{aligned} \tag{10}$$

If additionally we require $\hat{\mathcal{R}}_{2,1}\hat{\mathcal{R}}_{1,2} = \mathrm{Id}$, the algebra is called triangular. It is easy to check that for $q = 1$ $U_q(sl_N)$ becomes triangular with $\hat{\mathcal{R}}_{1,2} = P_{1,2}$.

These two equations connect a permutation between two reducible representations with permutations of irreducible representations. It effectively means that we need the only $\mathcal{R}$-matrix given for fundamental representation to express $\mathcal{R}$-matrix in any representation.

Let us write the action of $\mathcal{R}$-matrices on arbitrary representations $R_1 \otimes R_2 \otimes \cdots \otimes R_n$:

$$\check{\mathcal{R}}_{i,i+1} = \pi(\sigma_i) = id \otimes \cdots \otimes id \otimes P_{i,i+1}\mathcal{R}_{i,i+1} \otimes id \otimes \cdots \otimes id \tag{11}$$

and analogously $\check{\mathcal{R}}_{\rho(i)\rho(j)}$ for arbitrary permutation $\rho \in \mathbb{S}_n$ on $R_{\rho(1)} \otimes \cdots \otimes R_{\rho(n)}$. The operators $\check{\mathcal{R}}_i$ are invertible and associative, so they form a group. In fact, it is known as the representation of Artin braid group $\mathcal{B}_n$.

Definition 3. The Artin braid group $\mathcal{B}_n$ on n strands is a group generated by σ the following relations:

$$\begin{aligned} \sigma_i\sigma_j = \sigma_j\sigma_i, \qquad & \text{if } |i-j| > 1 \\ \sigma_i\sigma_j\sigma_i = \sigma_j\sigma_i\sigma_j, \qquad & \text{if } |i-j| = 1, \qquad i,j = 1,\dots,n \end{aligned} \tag{12}$$

The first property is trivially satisfied for $\check{\mathcal{R}}_{i,i+1} = \pi(\sigma_i)$. The second property is the braiding property and in quantum algebra it corresponds to the celebrated Yang-Baxter equation.

Definition 4. Yang-Baxter equation is the identity on $\mathcal{R}$-matrices:

$$\check{\mathcal{R}}_{1,2}\check{\mathcal{R}}_{1,3}\check{\mathcal{R}}_{2,3} = \check{\mathcal{R}}_{2,3}\check{\mathcal{R}}_{1,3}\check{\mathcal{R}}_{1,2} \tag{13}$$

As we will show below, this property follows from $\mathcal{R}$-matrix definition, but sometimes $\mathcal{R}$-matrix is defined as the solution to Yang-Baxter equation. This approach lead to a greater number of solutions, e.g. $\hat{\mathcal{R}}_{1,2} = P_{1,2}$ always satisfies the Yang-Baxter equation, but it does not form an quasitriangular Hopf algebra, unless $q = 1$ and it is triangular.

The eigenvalues of $\mathcal{R}$-matrix representation has an explicit form [3] in terms of quadratic Casimir eigenvalues κ. Given $\mathcal{R}_{1,2}$ acting in the space of $R_1 \otimes R_2 = \bigoplus_i Q_i$, where Q_i can coincide, the eigenvalues are expressed as:

$$\lambda_i(\mathcal{R}_{1,2}) = \pm q^{\kappa_{Q_i} - \kappa_{R_1} - \kappa_{R_2}}, \qquad \kappa(R_\mu) = \kappa(\mu) = \sum_{(i,j)\in\mu} (i-j) \tag{14}$$

where the sign depends on the choice of triple (R_1, R_2, Q_i). If $R_1 \neq R_2$, the sign is free to choose, but should be consistent with other signs that were chosen. If $R_1 = R_2$, the sign is fixed to be ± 1 depending on Q_i being from symmetric or antisymmetric part of the decomposition. See [11] for more details. In this paper we choose a convention that for both $R_1 = R_2$ and $R_1 \neq R_2$ we use the same rule. If we order all Q_i Young diagrams lexicographically, then the signs should alternate. The overall sign is chosen in a self-consistent way, that is, the $\mathcal{R}$-matrix axioms are satisfied.

1.2 General properties of multiplicity-free 6-j symbols

We recall that in this paper we assume that all 6-j symbols we work with are multiplicity-free except the opposite is stated. The expressions with multiplicities a bit differs and can be found in [7, 2, 12].

Orthogonality and tetrahedral symmetry. Racah matrices are unitary, consequently, 6-j symbols satisfy orthogonality relation [13]:

$$\sum_{R_{12}} \begin{Bmatrix} R_1 & R_2 & R_{12} \\ R_3 & R_{123} & R_{23} \end{Bmatrix} \overline{\begin{Bmatrix} R_1 & R_2 & R_{12} \\ R_3 & R_{123} & R'_{23} \end{Bmatrix}} D_{12}\sqrt{D_{23}D_{23'}} = \delta_{R_{23}R'_{23}} \cdot I, \qquad \begin{cases} I = 1 \text{ if symbols exist} \\ I = 0 \text{ in other case} \end{cases} \tag{15}$$

The presence of factor I is due to the fact that this relation should hold even if the left-hand side is trivial. If some of the representations does not respect fusion rules, then the left-had side becomes zero, implying $I=0$. One may choose arbitrary R_1, R_2, R_3, then the Racah matrix is non-trivial if and only if $R_4 \subset R_1 \otimes R_2 \otimes R_3$.

Definition 5. Tetrahedral symmetries are the symmetries between 6-j symbols [7] that are generated by relations

$$\begin{Bmatrix} R_1 & R_2 & R_{12} \\ R_3 & R_{123} & R_{23} \end{Bmatrix} = \begin{Bmatrix} \overline{R_3} & \overline{R_2} & \overline{R_{23}} \\ \overline{R_1} & \overline{R_{123}} & \overline{R_{12}} \end{Bmatrix} = \begin{Bmatrix} R_3 & \overline{R_{123}} & \overline{R_{12}} \\ R_1 & \overline{R_2} & \overline{R_{23}} \end{Bmatrix} = \begin{Bmatrix} R_2 & \overline{R_{12}} & \overline{R_1} \\ R_{123} & R_{23} & R_3 \end{Bmatrix} = \\ = \pm \begin{Bmatrix} R_2 & R_1 & R_{12} \\ \overline{R_{123}} & \overline{R_3} & \overline{R_{23}} \end{Bmatrix}, \tag{16}$$

where the $\pm$ phase should be consistently defined in order to satisfy the Biedenharn-Elliott identity.

These symmetries can be derived from the properties of 3j-symbols [2]. Phases arise as the sign of $\mathcal{R}$-matrix eigenvalues. See [7, 11, 14] for the discussion on phase conventions. One can check that these symmetries form a group of 24 elements isomorphic to $\mathbb{S}_4$, that is, the tetrahedron point group.

Racah back-coupling rule. One interesting property follows from the $\mathcal{R}$-matrix properties.

Definition 6. The Racah back-coupling rule is a general property of 6-j symbols [7]:

$$q^{\varkappa_a+\varkappa_b+\varkappa_c+\varkappa_{123}-\varkappa_{12}-\varkappa_{23}} \begin{Bmatrix} a & b & R_{12} \\ c & R_{123} & R_{23} \end{Bmatrix} = \sum_{R_{13}} \pm D_{13} q^{\varkappa_{13}} \begin{Bmatrix} b & a & R_{12} \\ c & R_{123} & R_{13} \end{Bmatrix} \begin{Bmatrix} a & c & R_{13} \\ b & R_{123} & R_{23} \end{Bmatrix} \tag{17}$$

Above we use the notations a, b, c for representations only to clarify the structure of equation.

Proposition 1. *Racah back-coupling rule is equivalent to hexagon axioms of $\mathcal{R}$-matrix.*

Proof. The property can be proven using the $\mathcal{R}$-matrix definition. Let us consider the first of the hexagon axioms:

$$\check{\mathcal{R}}_{12,3} = \check{\mathcal{R}}_{1,3}\check{\mathcal{R}}_{2,3} \tag{18}$$

We recall that from the spectral decomposition of $\mathcal{R}$-matrices $\mathcal{R}_{2,3}$ acts as a constant map in each subspace V_i^{23} from the space $V_2 \otimes V_3 = \bigoplus_i M_i^{23} \otimes V_i^{23}$. If we choose the basis of the highest weight, $\mathcal{R}$-matrix becomes block-diagonal, where different blocks correspond to different representations Y_i with block size equal to multiplicity $\dim M_i^{23}$. Moreover, we can make it diagonal by the specification of basis in multiplicity space, then each block takes the form of unit matrix times the eigenvalue given by (14).

Let us now consider the whole space $V_1 \otimes V_2 \otimes V_3$, in which $id \otimes \mathcal{R}_{2,3}$ acts. We restrict ourselves to the particular representation $R_4 \subset R_1 \otimes R_2 \otimes R_3$ without loss of generality. The corresponding subspace is $\bigoplus_i M_4^{1i} \otimes M_i^{23} \otimes V_{123}$, where the overall set of non-trivial i is such that $M_i^{23} \neq \varnothing \neq M_4^{1i}$. We can consider a Racah matrix that acts on this subspace, transforming the highest weight basis of $(R_1 \otimes R_2) \otimes R_3$ into $R_1 \otimes (R_2 \otimes R_3)$. Thus, the $\mathcal{R}_{1,2}$ eigenbasis transforms under the unitary rotation into the eigenbasis of $\mathcal{R}_{2,3}$.

The operator $\mathcal{R}_{12,3}$ is trickier as we should understand it as $\mathcal{R}_{12,3} = \bigoplus_i \mathcal{R}_{i3}$, where i is enumerating all irreducible representations from $R_1 \otimes R_2 = M_i^{12} \otimes X_i$. In this sense our restriction to the R_4 makes each $\mathcal{R}_{i3}$ just a constant map and possible rotation in the multiplicity space, whereas the whole $\mathcal{R}_{12,3}$ is again block-diagonal and depends on X_i. It makes $\mathcal{R}_{12,3}$ and $\mathcal{R}_{1,2}$ very similar. In fact, if we compare their eigenvalues, we make a conclusion that:

$$\mathcal{R}_{12,3} = q^{\varkappa_4-\varkappa_1-\varkappa_2-\varkappa_3} \cdot \mathcal{R}_{1,2}^{-1} \otimes id \tag{19}$$

If we decompose each matrix $\mathcal{R}_{a,b} = W\widetilde{\mathcal{R}}_{a,b}U$, where W, U are unitary and $\widetilde{\mathcal{R}}_{a,b}$ is diagonal. Then the equation becomes:

$$\check{\widetilde{\mathcal{R}}}_{1,2}U_{312}\check{\widetilde{\mathcal{R}}}_{1,3}U_{132}^{\dagger}\check{\widetilde{\mathcal{R}}}_{2,3}U_{123} = P_{123} \cdot q^{\varkappa_4-\varkappa_1-\varkappa_2-\varkappa_3}, \qquad P_{123}((v_1 \otimes v_2) \otimes v_3) := v_3 \otimes (v_2 \otimes v_1) \tag{20}$$

We can see that unitary matrices are in fact Racah matrices with identification $U_{123} = U\begin{pmatrix} R_1 & R_2 \\ R_3 & R_4 \end{pmatrix}$, the same for U_{132}, U_{312}. If we normalize the equation, write it in terms of 6-j symbols and use tetrahedral symmetry $U_{123} = U_{321}^{\dagger}$, we will get exactly the Racah back-coupling rule. □

Corollary 1. *Racah back-coupling rule implies the Yang-Baxter equation.*

Proof. The other equation from $\mathcal{R}$-matrix definition, namely

$$\check{\mathcal{R}}_{1,23} = \check{\mathcal{R}}_{1,3}\check{\mathcal{R}}_{1,2} \tag{21}$$

leads to the similar condition on Racah matrices:

$$U_{321}\check{\widetilde{\mathcal{R}}}_{2,3}U_{231}^{\dagger}\check{\widetilde{\mathcal{R}}}_{1,3}U_{213}\check{\widetilde{\mathcal{R}}}_{1,2} = P_{123} \cdot q^{\varkappa_4-\varkappa_1-\varkappa_2-\varkappa_3} \tag{22}$$

This will lead to the same back-coupling rule as (20).

Normalized or not, equations (20) and (22) have identical right-hand side and thus we can write:

$$\check{\widetilde{\mathcal{R}}}_{1,2}U_{312}\check{\widetilde{\mathcal{R}}}_{1,3}U_{132}^{\dagger}\check{\widetilde{\mathcal{R}}}_{2,3}U_{123} = U_{321}\check{\widetilde{\mathcal{R}}}_{2,3}U_{231}^{\dagger}\check{\widetilde{\mathcal{R}}}_{1,3}U_{213}\check{\widetilde{\mathcal{R}}}_{1,2} \tag{23}$$

which is exactly the Yang-Baxter equation. As both (20) and (22) are equivalent up to a normalization and the Yang-Baxter equation is homogeneous, the Racah back-coupling rule leads to the Yang-Baxter equation. The opposite is not true. □

Pentagon relation. The following condition on 6-j symbol is central for our discussion. It does not involve $\mathcal{R}$-matrices and in fact more general. If we consider the product of four representations, there are five ways to decompose them:

$$\begin{aligned} &((R_1 \otimes R_2) \otimes R_3) \otimes R_4 \\ &(R_1 \otimes (R_2 \otimes R_3)) \otimes R_4 \\ &(R_1 \otimes R_2) \otimes (R_3 \otimes R_4) \\ &R_1 \otimes ((R_2 \otimes R_3) \otimes R_4) \\ &R_1 \otimes (R_2 \otimes (R_3 \otimes R_4)) \end{aligned} \tag{24}$$

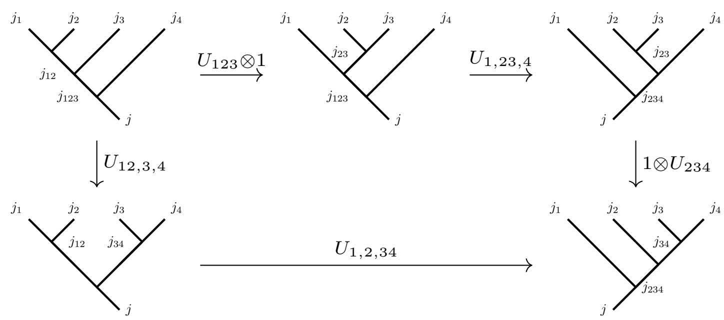
These decompositions correspond to five different bases, which can be transformed one into another by Racah matrix. The new idea here is that the transformation can be done in two distinct ways. Let us consider for example the transformation:

$$W : ((R_1 \otimes R_2) \otimes R_3) \otimes R_4 \to R_1 \otimes (R_2 \otimes (R_3 \otimes R_4)) \tag{25}$$

Then W can be written ether as W' or as W'':

$$
\begin{aligned}
W' &= (1 \otimes U_{234}) U_{1,23.4} (U_{123} \otimes 1) \\
W'' &= U_{1,2,34} U_{12.3.4}
\end{aligned}
\tag{26}
$$

As the basis is determined by the product order uniquely, we should have the equality $W = W'$. The corresponding commutative diagram can be written with a tree denoting the basis in the multiplicity space:



This identity is known as pentagon equation, or, specifically for 6-j symbols, it is called after Biedenharn and Elliott.

Definition 7. The generalized Biedenharn-Elliott identity, or pentagon identity [15] is

$$
\sum_{R_{23}} D_{23} \begin{Bmatrix} R_1 & R_2 & R_{12} \\ R_3 & R_{123} & R_{23} \end{Bmatrix} \begin{Bmatrix} R_1 & R_{23} & R_{123} \\ R_4 & R_{1234} & R_{234} \end{Bmatrix} \begin{Bmatrix} R_2 & R_3 & R_{23} \\ R_4 & R_{234} & R_{34} \end{Bmatrix} = \\
= \begin{Bmatrix} R_{12} & R_3 & R_{123} \\ R_4 & R_{1234} & R_{34} \end{Bmatrix} \begin{Bmatrix} R_1 & R_2 & R_{12} \\ R_{34} & R_{1234} & R_{234} \end{Bmatrix}
\tag{27}
$$

This equation has a lot of nice properties, for example:

Proposition 2. *The pentagon identity imply the 6-j symbols orthogonality relation.*

Proof. We set R_{1234} to be a trivial representation and denote its Young diagram as $[\![0]\!]$. The expression is either of the form $0 = 0$ or non-trivial depending on other representations. The only non-trivial equation is present when $R_{123} = \overline{R_4}$ and $R_{234} = \overline{R_1}$. We additionally denote R_{34} as $\overline{R'_{12}}$.

$$
\sum_{R_{23}} D_{23} \begin{Bmatrix} R_1 & R_2 & R_{12} \\ R_3 & \overline{R_4} & R_{23} \end{Bmatrix} \begin{Bmatrix} R_1 & R_{23} & \overline{R_4} \\ R_4 & [\![0]\!] & \overline{R_1} \end{Bmatrix} \begin{Bmatrix} R_2 & R_3 & R_{23} \\ R_4 & \overline{R_1} & \overline{R'_{12}} \end{Bmatrix} = \\
= \begin{Bmatrix} R_{12} & R_3 & \overline{R_4} \\ R_4 & [\![0]\!] & \overline{R'_{12}} \end{Bmatrix} \begin{Bmatrix} R_1 & R_2 & R_{12} \\ \overline{R'_{12}} & [\![0]\!] & \overline{R_1} \end{Bmatrix}
\tag{28}
$$

6-j symbols with one trivial representation correspond to unitary matrix of dimension 1 and called trivial 6-j symbols. All of them are ± 1 up to a normalization.

$$
\begin{Bmatrix} R_1 & R_2 & R_{12} \\ \overline{R'_{12}} & [\![0]\!] & \overline{R'_1} \end{Bmatrix} = \frac{\pm \delta_{R_{12} R'_{12}} \delta_{R_1 R'_1}}{\sqrt{D_1 D_4}}
\tag{29}
$$

Substitution of trivial 6-j symbols into pentagon give us:

$$\sum_{R_{23}} D_{23} \begin{Bmatrix} R_1 & R_2 & R_{12} \\ R_3 & \overline{R_4} & R_{23} \end{Bmatrix} \begin{Bmatrix} R_2 & R_3 & R_{23} \\ R_4 & \overline{R_1} & \overline{R'_{12}} \end{Bmatrix} = \frac{\pm\delta_{R_{12}R'_{12}}}{\sqrt{D_{12}D'_{12}}} \tag{30}$$

which becomes the orthogonality relation if we perform tetrahedral symmetry. □

Summary. We have stated the main properties of 6-j symbols in ths subsection. They can be divided into symmetries and identities. For arbitrary 6-j symbol only tetrahedral symmetries are known to hold. Two main identities are the pentagon identity and the Racah back-coupling rule. The Yang-Baxter equation and the orthogonality can be deduced from the main two properties:

1. Pentagon equation ⇒ Orthogonality relation

2. Racah back-coupling rule ⇒ Yang-Baxter equation

Interestingly, the first and the second lines are very different. There is no obvious way to connect these properties. The first line is defined without $\mathcal{R}$-matrix. The pentagon identity is a very strong condition on 6-j symbols. However, it can not determine the 6-j symbol by itself, but rather recursively expand it. The second line is not a strong constraint on 6-j symbols itself, one may compare the number of variables in the equation with the number of equation components. It can not fix Racah matrices of arbitrary size even with orthogonality condition imposed.

We combine these two identities to determine the 6-j symbol. Roughly speaking, we solve the pentagon identity in terms of primitive 6-j symbols for a wide class of 6-j symbols. Then we show that using the Racah back-coupling rule to determine the primitive 6-j symbols. In particular, we solve the pentagon identity in the class of symmetric and conjugate to symmetric representations.

1.3 q-Hypergeometric series and Racah polynomial

There is a strong connection between 6-j symbols, q-hypergeometric series and orthogonal polynomials. That is, properly normalized 6-j symbols at least for $U_q(sl_2)$ can be expressed as a terminating q-hypergeometric series. Moreover, it is equal to a q-Racah polynomial. This property singles out 6-j symbols among other objects and it can be used for calculations as orthogonal polynomial always can be defined recursively.

In this subsection we recall the notion of q-hypergeometric series [16] and some essential facts from the theory of orthogonal polynomials.

q-Hypergeometric series. A q-Pochhammer symbol is defined as $(a,q)_n = \prod_{k=0}^{n-1}(1-aq^k)$, which is closely related to quantum factorial $[a]! = \prod_{i=1}^{a}[a]$.

Definition 8. The q-hypergeometric series are defined as:

$${}_{p+1}\phi_p \left(\begin{matrix} a_1, \dots, a_{p+1} \\ b_1, \dots, b_p \end{matrix} ; q, z \right) := \sum_{n=0}^{\infty} \frac{(a_1,q)_n \dots (a_{p+1},q)_n}{(b_1,q)_n \dots (b_p,q)_n (q,q)_n} z^n, \tag{31}$$

or, alternatively, it can be also written as:

$${}_{p+1}\Phi_p \left(\begin{matrix} a_1, \dots, a_p, a_{p+1} \\ b_1, \dots, b_p \end{matrix} ; q, z \right) := {}_{p+1}\phi_p \left(\begin{matrix} q^{a_1}, \dots, q^{a_p}, q^{a_{p+1}} \\ q^{b_1}, \dots, q^{b_p} \end{matrix} ; q, z \right). \tag{32}$$

It is far more convenient because it may be reformulated in terms of q-factorials:

$$ {}_{p+1}\Phi_p\left(\begin{matrix} a_1+1,\dots,a_p+1,a_{p+1}+1 \\ b_1+1,\dots,b_p+1 \end{matrix};q,z\right) = \sum_{n=0}^{\infty} \frac{[a_1+n]!}{[a_1]!}\dots\frac{[a_{p+1}+n]!}{[a_{p+1}]!}\frac{[b_1]!}{[b_1+n]!}\dots\frac{[b_p]!}{[b_p+n]!}\frac{z^n}{[n]!}. \tag{33} $$

This expression evidently has the limit $\lim\limits_{q\to 1}[a]! = a!$, where the whole series becomes a usual hypergeometric function.

There are a lot of hypergeometric function symmetries generated by the following property.

Definition 9. Permutation symmetry is the evident property of ${}_r\Phi_p$ functions to be invariant under permutations $\omega \in \mathbb{S}_r$ and $u \in \mathbb{S}_p$:

$$ {}_r\Phi_p\left(\begin{matrix} a_1,\dots,a_r \\ b_1,\dots,b_p \end{matrix};q,z\right) = {}_r\Phi_p\left(\begin{matrix} a_{\omega(1)},\dots,a_{\omega(r)} \\ b_{u(1)},\dots,b_{u(p)} \end{matrix};q,z\right). \tag{34} $$

The ${}_{r+1}\Phi_r$ series with the constraint on the sum of arguments $1+\sum\limits_{i=1}^{r+1} a_i = \sum\limits_{i=1}^{r} b_i$ and with $z=q$ are called Saalschützian one. There is a particular series we are are interested a lot, namely, Saalschützian ${}_4\Phi_3$.

Definition 10. Sears' transformation [16] is the relation between two Saalschützian ${}_4\Phi_3$ functions:

$$ {}_4\Phi_3\left(\begin{matrix} x,y,z,n \\ u,v,w \end{matrix};q,q\right) = \frac{[v-z-n-1]![u-z-n-1]![v-1]![u-1]!}{[v-z-1]![v-n-1]![u-z-1]![u-n-1]!}\,{}_4\Phi_3\left(\begin{matrix} w-x,w-y,z,n \\ 1-u+z+n,1-v+z+n,w \end{matrix};q,q\right), \tag{35} $$

where $x+y+z+n+1 = u+v+w$.

For Saalschützian ${}_4\Phi_3$ both permutation symmetry and Sears' transformation form a large group of symmetries [17, 6].

6-j symbol known expressions. The 6-j symbols are known to be connected with q-hypergeometric series at least for $U_q(sl_2)$ algebra [18]. In this case each argument of 6-j symbol is a one-row Young diagram which is parametrized by a non-negative integer. The correspondence follows form the following result [4].

Proposition 3. *Arbitrary $U_q(sl_2)$ 6-j symbol up to a monomial factor is equal to a terminating Saalschützian q-hypergeometric series ${}_4\Phi_3$:*

$$ \begin{Bmatrix} [\![s_1]\!] & [\![s_2]\!] & [\![i]\!] \\ [\![s_3]\!] & [\![s_4]\!] & [\![j]\!] \end{Bmatrix} = K\cdot {}_4\Phi_3\left(\begin{matrix} a_1,a_2,a_3,a_4 \\ b_1,b_2,b_3 \end{matrix};q,q\right), \quad 2a_i = \begin{pmatrix} -s_1-s_2+i \\ -s_3-s_4+i \\ -s_1-s_4+j \\ -s_2-s_3+j \end{pmatrix}, 2b_i = \begin{pmatrix} -s_1-s_2-s_3-s_4-2 \\ i+j-s_2-s_4+2 \\ i+j-s_1-s_3+2 \end{pmatrix}. \tag{36} $$

and the monomial factor is:

$$ K = \frac{\theta(s_1,s_2,i)\,\theta(s_3,s_4,i)\,\theta(s_1,s_4,j)\,\theta(s_2,s_3,j)\,[\frac{s_1+s_2+s_3+s_4}{2}+1]!}{[\frac{s_3+s_4-i}{2}]![\frac{s_1+s_2-i}{2}]![\frac{s_2+s_3-j}{2}]![\frac{s_1+s_4-j}{2}]![\frac{i+j-s_2-s_4}{2}]![\frac{i+j-s_1-s_3}{2}]!}, \quad \theta(a,b,c) := \sqrt{\frac{[\frac{a+b-c}{2}]![\frac{c+a-b}{2}]![\frac{b+c-a}{2}]!}{[\frac{a+b+c}{2}+1]!}}. \tag{37} $$

There are Racah matrices in $U_q(sl_N)$ with only symmetric and conjugate to symmetric R_1, R_2, R_3 representations in arguments. The corresponding 6-j symbols will have a relatively simple form. There are two types of such 6-j symbols we are interested the most.

$$\text{type I}^+ : \begin{Bmatrix} [\![s_1]\!] & \overline{[\![s_2]\!]} & R_{12} \\ [\![s_3]\!] & [\![s_4, c_4^{N-2}]\!] & R_{23} \end{Bmatrix} \qquad \text{type II}^+ : \begin{Bmatrix} [\![s_1]\!] & [\![s_2]\!] & R_{12} \\ \overline{[\![s_3]\!]} & [\![s_4, c_4^{N-2}]\!] & R_{23} \end{Bmatrix} \tag{38}$$

Here one may see that the family of possible R_{12} and R_{23} is enumerated by one parameter. From fusion rules it is easy to derive for the first type $R_{12}^{(i)} = [\![i, \frac{s_2-s_1+i}{2}^{N-2}]\!]$, $R_{23}^{(j)} = [\![j, \frac{s_2-s_3+j}{2}^{N-2}]\!]$. There is a linear relation on type I^+ arguments from fusion rules $s_1+s_3-s_2-s_4 = -2c_4$, we choose to work with arbitrary s_i and assume that c_4 satisfy this relation. The special case when $c_4 = 0$ is called type I and correspond to a simpler situation when the fourth representation is symmetric [7]. Similarly, the second type has $R_{12}^{(i)} = [\![\frac{s_1+s_2+i}{2}, \frac{s_1+s_2-i}{2}]\!]$ and $R_{23}^{(j)} = [\![j, \frac{s_2-s_3+j}{2}^{N-2}]\!]$. The restrictions require equalities:

$$\begin{aligned} &s_1+s_3-s_2-s_4 = -2c_4 \quad \text{for type I}^+, \qquad s_1+s_3 = s_2+s_4 \quad \text{for type I}, \\ &s_1+s_2-s_3-s_4 = -2c_4 \quad \text{for type II}^+, \qquad s_1+s_2 = s_3+s_4 \quad \text{for type II}. \end{aligned} \tag{39}$$

We use these classes of symbols a lot in this work, so it is convenient to improve our notation specifically for these two types. The expressions for them are almost the same, so we denote 6-j symbol of type I and I^+ as:

$$\begin{bmatrix} s_1 & s_2 & i \\ s_3 & s_4 & j \end{bmatrix}_1 := \begin{Bmatrix} [\![s_1]\!] & \overline{[\![s_2]\!]} & \left[\!\!\left[i, \frac{s_2-s_1+i}{2}^{N-2}\right]\!\!\right] \\ [\![s_3]\!] & [\![s_4, c_4^{N-2}]\!] & \left[\!\!\left[j, \frac{s_2-s_3+j}{2}^{N-2}\right]\!\!\right] \end{Bmatrix}, \tag{40}$$

and type II, II^+:

$$\begin{bmatrix} s_1 & s_2 & i \\ s_3 & s_4 & j \end{bmatrix}_2 := \begin{Bmatrix} [\![s_1]\!] & [\![s_2]\!] & \left[\!\!\left[\frac{s_1+s_2+i}{2}, \frac{s_1+s_2-i}{2}\right]\!\!\right] \\ \overline{[\![s_3]\!]} & [\![s_4, c_4^{N-2}]\!] & \left[\!\!\left[j, \frac{s_2-s_3+j}{2}^{N-2}\right]\!\!\right] \end{Bmatrix}, \tag{41}$$

where i, j are defined in such a way in order to have a nice $N = 2$ limit.

The expression for type I and type II depends on q and 7 integer parameters, i.e. 6 for representations and N. The answer was conjectured in [1] and was later written in terms of Saalschützian ${}_4\Phi_3$ [6].

Conjecture 1. *Multiplicity-free $U_q(sl_N)$ 6-j symbol expression is given by the formula [1, 6] is*

$$\begin{bmatrix} r_1 & r_2 & i \\ r_3 & r_4 & j \end{bmatrix}_T = K_T \cdot {}_4\Phi_3 \left(\begin{matrix} a_1, a_2, a_3, a_4 \\ b_1, b_2, b_3 \end{matrix}; q, q \right), \tag{42}$$

$$2a_i = \begin{pmatrix} -r_1 - r_2 + i - 2(N-2)\delta_{T,2} \\ -r_3 - r_4 + i \\ -r_1 - r_4 + j \\ -r_2 - r_3 + j \end{pmatrix}, \qquad 2b_i = \begin{pmatrix} -r_1 - r_2 - r_3 - r_4 - 2(N-1) \\ i + j - r_2 - r_4 + 2 \\ i + j - r_1 - r_3 + 2 + 2(N-2)\delta_{T,1} \end{pmatrix}. \tag{43}$$

where $r_1 + r_3 = r_2 + r_4$ for type I and $r_1 + r_2 = r_3 + r_4$ for type II. The factor is given by

$$K_T = \frac{\theta_N(r_1, r_2, i)\,\theta_N(r_3, r_4, i)\,\theta_N(r_1, r_4, j)\,\theta_N(r_2, r_3, j)\,[N-1]_q![N-2]_q![\frac{r_1+r_2+r_3+r_4}{2} + N - 1]_q!}{[\frac{r_3+r_4-i}{2}]_q![\frac{r_1+r_2-i}{2} + (N-2)\delta_{T,2}]_q![\frac{r_2+r_3-j}{2}]_q![\frac{r_1+r_4-j}{2}]_q![\frac{i+j-r_2-r_4}{2}]_q![\frac{i+j-r_1-r_3}{2} + (N-2)\delta_{T,1}]_q!}. \tag{44}$$

All considered examples confirm this expression. The aim of this paper is to prove this formula. Also the developed method allow us to obtain an expression for types I^+ and II^+ in terms of q-Racah polynomial.

Orthogonal polynomials.

Definition 11. q-Racah polynomials are a set of orthogonal polynomials [19] defined as a Saalschützian q-

hypergeometric series ${}_4\Phi_3$:

$$\mathfrak{R}_n(\nu(x);\alpha,\beta,\gamma,\delta|q) = {}_4\Phi_3\left(\begin{matrix}-n, n+\alpha+\beta+1, -x, x+\gamma+\delta+1\\ \alpha+1, \beta+\delta+1, \gamma+1\end{matrix}; q, q\right) \qquad n = 0, 1, \dots L \tag{45}$$

where n is the degree of the polynomial in $\nu(x) := q^{-x} + q^{\gamma+\delta+x+1}$, L is non-negative and have on of the three possible values:

$$L = \begin{bmatrix} -\alpha-1 \\ -\beta-\delta-1 \\ -\gamma-1 \end{bmatrix} \tag{46}$$

The condition of L non-negativity implies that the series is terminating. It is indeed a polynomial in $\nu(x)$ as the dependence on x is:

$$(q^{-x}, q^{x+\gamma+\delta+1}; q)_x = \prod_{j=0}^{k-1}(1 - \nu(x)q^j + q^{\gamma+\delta+2j+1}) \tag{47}$$

From the equation for 6-j symbol in $U_q(sl_2)$ it is possible to relate it with a q-Racah polynomial, because the terminating Saalschützian ${}_4\Phi_3$ series is in fact the Racah polynomial. In particular, orthogonality of 6-j symbols and Racah polynomials is the same equality written in different terms. Other properties also can be translated between these two objects.

As orthogonal polynomials, they obey the following set of identities. For brevity we omit parameters $\alpha, \beta, \gamma, \delta, q$ in $\mathfrak{R}_n(\nu(x))$. By convention, in a recurrence relations $\mathfrak{R}_{-1}(\nu(x))$ is set to 0.

- Three-term recurrence relation (3TRR):

$$[x][x+\gamma+\delta+1]\mathfrak{R}_n(\nu(x)) = A_n\mathfrak{R}_{n+1}(\nu(x)) - (A_n + C_n)\mathfrak{R}_n(\nu(x)) + C_n\mathfrak{R}_{n-1}(\nu(x)) \tag{48}$$

  with coefficients specified for q-Racah polynomial:

$$\begin{aligned} A_n &= \frac{[n+\alpha+1][n+\alpha+\beta+1][n+\beta+\delta+1][n+\gamma+1]}{[2n+\alpha+\beta+1][2n+\alpha+\beta+2]} \\ C_n &= \frac{[n][n+\alpha+\beta-\gamma][n+\alpha-\delta][n+\beta]}{[2n+\alpha+\beta][2n+\alpha+\beta+1]} \end{aligned} \tag{49}$$

- Orthogonality relation:

$$\begin{gathered} \sum_{x=0}^{L} \frac{(q^{\alpha+1}, q^{\beta+\delta+1}, q^{\gamma+1}, q^{\gamma+\delta+1}; q)_x}{(q, q^{\gamma-\alpha+\delta+1}, q^{\gamma-\beta+1}, q^{\delta+1}; q)_x} \frac{[\gamma+\delta+2x+1]}{q^{x(\alpha+\beta)}[\gamma+\delta+1]}\mathfrak{R}_n(\nu(x))\mathfrak{R}_m(\nu(x)) = h_n\delta_{mn} \\ h_n = \frac{(q^{\gamma-\alpha-\beta}, q^{\delta-\alpha}, q^{-\beta}, q^{\gamma+\delta+2}; q)_\infty}{(q^{-\alpha-\beta-1}, q^{\gamma-\alpha+\delta+1}, q^{\gamma-\beta+1}, q^{\delta+1}; q)_\infty} \frac{[\alpha+\beta+1]q^{n(\gamma+\delta)}}{[\alpha+\beta+2n+1]} \frac{(1, q^{\alpha+\beta-\gamma+1}, q^{\alpha-\delta+1}, q^{\beta+1}; q)_n}{(q^{\alpha+1}, q^{\alpha+\beta+1}, q^{\beta+\delta+1}, q^{\gamma+1}; q)_n} \end{gathered} \tag{50}$$

Proposition 4. *Each orthogonal polynomial sequence possess a three-term recurrence relation.*

The proof of this fact is rather simple and can be found in the literature. Let us assume we work with orthogonal discrete polynomials $p_n(x)$ defined on $0 \le x \le N-1$ for $0 \le n \le N-1$, $x, n \in \mathbb{Z}$. The naive estimation of a free parameters in this system from the orthogonality is $\frac{N(N-1)}{2}$. However, as it can be seen, a three-term recurrence relation allow us to use only $3N$ coefficients, which does not depend on x, to fully determine the polynomials.

The three-term recurrence relation can help a lot if we do not know the polynomials explicitly, but we can obtain the 3TRR on them and it is identical to some known polynomial's 3TRR. Obviously, the set of polynomials is the same in this case. It is reasonable to ask whether the orthogonality relation is the same or not. The important result is that the orthogonality measure is unique [19].

The monic polynomial is a polynomial $y_n(x) = \sum_{n=0}^{i} k_i x^i$ with the normalization condition $k_n = 1$.

Theorem 1 (Favard's Theorem)**.** *Let y_n denote the monic polynomial of degree $n \in \{0, 1, 2, \dots\}$ satisfying the three-term recurrence relation*

$$y_{n+1}(x) = (x - c_n) y_n(x) - d_n y_{n-1}(x), \qquad c_n, d_n \in \mathbb{C}, \quad n = 1, 2, 3, \dots \tag{51}$$

Then there exists a unique linear functional Λ with

$$\Lambda(1) = 1, \qquad \Lambda(y_n y_m) = 0, \qquad \textit{for } n \neq m, \quad m, n \in \{0, 1, 2, \dots\} \tag{52}$$

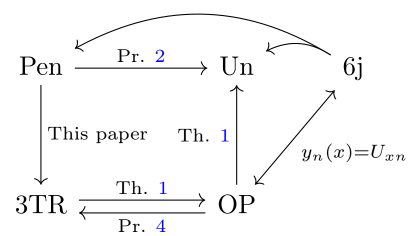
This linear functional Λ is quasi-definite if and only if $d_n = 0$ for all $n = 1, 2, 3, \dots$. This linear functional Λ is positive-definite if and only if $c_n \in \mathbb{R}$ for all $n = 0, 1, 2, \dots$ and $d_n > 0$ for all $n = 1, 2, 3, \dots$.

1.4 Relation between 6-j symbols and orthogonal polynomials

In this paper we show that multiplicity-free 6-j symbols possess a three-term recurrence relation. The derivation only uses pentagon identity and fusion rules. This derivation allow one to say a lot about 6-j symbols.

We can illustrate the relationship between different objects by the following diagram.

- (Pen) Pentagon equation
- (Un) Racah matrix unitarity, 6-j symbol orthogonality
- (3TR) Three-term recurrence relation
- (OP) Orthogonal polynomial sequence
- (6j) Expression for arbitrary 6-j symbols



In the upper row we place the properties of 6-j symbols coming from the Hopf algebra. We have already shown that 6-j symbols are always orthogonal and satisfy pentagon equation. Due to the Proposition 2 any pentagon equation solution implies orthogonality.

The lower row is about orthogonal polynomials, the main result here is the Favard's theorem. It guarantees that the three-term recurrence relation uniquely determines not only polynomials but also the orthogonality measure. This orthogonality relation coincides with 6-j symbol's one due to the uniqueness. It is very helpful, because this identifies the abstract orthogonality from the theorem with 6-j symbols which are orthogonal in a very clear sense.

This identification between 6-j symbols and orthogonal polynomials allow us to write a diagonal arrow, although we do not know the expression for 6-j symbols. We now may ask what is the source of three-term recurrence relation on 6-j symbols. For $U_q(sl_2)$ it was shown that the pentagon equation can be reduced to the recurrent relation and solved explicitly [18, 2]. In the case of $U_q(sl_N)$ the pentagon equation is much harder to handle due to the not obvious structure of representation tensor products.

In this paper we fill this gap and write the three-term relation on the class of $U_q(sl_N)$ symbols with two symmetric and one rectangular representation. Corresponding to the written diagram above, we can start with the abstract pentagon equation and derive 6-j symbols in a unique way. Thus, explicit 6-j symbol formula is unique if the coefficients of 3TRR are obtained in a unique way. We can state three main achieved results.

- We proved that the wide class of multiplicity-free 6-j symbols are orthogonal polynomials with a certain three-term relation.
- This also implies that the pentagon equation has a unique solution.
- For a lesser subclass we get three-term relation and solve it, obtaining the explicit form for 6-j symbols in terms of Racah polynomials.

2 Three-term recurrence relation for 6-j symbols

2.1 Three-term relation for 6-j symbols with rectangular representation

Let us clarify the notation. We denote by J the representation that is from one of the three classes: symmetric $[\![s]\!]$, conjugate to symmetric $[\![s^{N-1}]\!]$ or antisymmetric $[\![1^k]\!]$ for $s \geq 0,\ k \geq 0$. We write symmetric representations as S. Rectangular representation $[\![s^k]\!]$ is denoted by T. We use $\varepsilon^{\pm}$ for either fundamental or antifundamental representation, that is $\varepsilon^+ = [\![1]\!]$, $\varepsilon^- = [\![1^{N-1}]\!]$.

Lemma 1. *Consider a product of representations $T_2 \otimes J_3 \otimes \varepsilon^{\pm}$, which is decomposed into $R_{23} \subset T_2 \otimes J_3$, $J_{34} \subset J_3 \otimes \varepsilon^{\pm}$ and $R_{234} \subset T_2 \otimes J_3 \otimes \varepsilon^{\pm}$. Fixing all representations except R_{23} there are only two possible representations for R_{23} in the product.*

Proof. All possible sequences of decompositions of the triple product $T_2 \otimes J_3 \otimes \varepsilon^{\pm}$ can be written as follows:

$$\begin{array}{ccccc} T_2 & & J_3 & & \varepsilon^{\pm} \\ & R_{23} & & J_{34} & \\ & & R_{234} & & \end{array} \qquad \begin{array}{c} R_{23} \subset T_2 \otimes J_3,\, J_{34} \subset J_3 \otimes \varepsilon^{\pm} \\ R_{234} = (R_{23} \otimes \varepsilon^{\pm}) \cap (T_2 \otimes J_{34}) \end{array} \tag{53}$$

The general form of R_{234} as a representation from $T_2 \otimes J_{34}$ can be written as $[\![a, b^k, c]\!]$, $[\![b^k, a, c^{N-k-2}]\!]$ or $[\![(b+1)^a, b^{k-a}, 1^c]\!]$ for J_{34} being symmetric, conjugate to symmetric or antisymmetric. Considering the case $\varepsilon^{\pm} = \varepsilon^-$ we find R_{23} as the decomposition of $R_{234} \otimes \varepsilon^+$. Let us decompose the product $R_{234} \otimes \varepsilon^+$ in each case using Littlewood-Richardson rules.

$$\begin{aligned} &[\![a, b^k, c]\!] \otimes [\![1]\!] = [\![a+1, b^k, c]\!] \oplus [\![a, b^k, c+1]\!] \oplus \underline{[\![a, b+1, b^{k-1}, c]\!]} \oplus \underline{[\![a, b^k, c, 1]\!]} \\ &[\![b^k, a, c^{N-k-2}]\!] \otimes [\![1]\!] = [\![b^k, a+1, c^{N-k-2}]\!] \oplus [\![(b-1)^k, a-1, (c-1)^{N-k-2}]\!] \oplus \\ &\qquad \oplus \underline{[\![b+1, b^{k-1}, a, c^{N-k-2}]\!]} \oplus \underline{[\![b^k, a, c+1, c^{N-k-3}]\!]} \\ &[\![(b+1)^a, b^{k-a}, 1^c]\!] \otimes [\![1]\!] = [\![(b+1)^{a+1}, b^{k-a-1}, 1^c]\!] \oplus [\![(b+1)^a, b^{k-a}, 1^{c+1}]\!] \oplus \\ &\qquad \oplus \underline{[\![b+2, (b+1)^{a-1}, b^{k-a}, 1^c]\!]} \oplus \underline{[\![(b+1)^a, b^{k-a}, 2, 1^{c-1}]\!]} \end{aligned} \tag{54}$$

Note that we have decomposed $R_{234} \otimes \varepsilon^+ \subset (T_2 \otimes J_{34}) \otimes \varepsilon^+$, but the decomposition for $T_2 \otimes J_3 \subset T_2 \otimes (J_{34} \otimes \varepsilon^+)$ is different. We underline the terms that can not lie in the decomposition $T_2 \otimes J_3$. The lasting terms are in general lead to non-trivial representation with corresponding 6-j symbols. In degenerate cases, e.g. $a = b$ or $k = 0$ some of the irreducible components may coincide or not exist. However, the number of possible terms is not greater than two. The product $R_{234} \otimes \varepsilon^-$ is analogous. One also can express it via the conjugation as $\overline{R_{234} \otimes \varepsilon^-} = \overline{R_{234}} \otimes \varepsilon^+$. One can see from the general form of R_{234} that $\overline{R_{234}}$ lie in the same class of representations. Thus, $R_{234} \otimes \varepsilon^{\pm}$ always produce at most two R_{23} that is compatible with iterative 6-j symbol. □

Definition 12. 6-j symbols with one (anti-)fundamental representation $\varepsilon^{\pm}$ are called the primitive ones. All of them can be written in the form

$$\begin{Bmatrix} R_1 & R_{23} & R_{123} \\ \varepsilon^{\pm} & R_{1234} & R_{234} \end{Bmatrix} \tag{55}$$

Lemma 2. *Pentagon equation implies a recurrence on 6-j symbols of the form*

$$\begin{Bmatrix} R_1 & R_2 & R_{12} \\ R_3 & R_{123} & R_{2345} \end{Bmatrix} = \sum_i C_i \begin{Bmatrix} R_1 & R_2 & R_{12} \\ R_3 & R_{123} & R_{23}^{(i)} \end{Bmatrix} \tag{56}$$

where C_i are some combinations of primitive 6-j symbols.

Proof. The pentagon relation in a general form is written in (27). After specifying $R_4 = \varepsilon^{\pm}$ it reads:

$$\begin{Bmatrix} R_1 & R_2 & R_{12} \\ R_{34} & R_{1234} & R_{234} \end{Bmatrix} \begin{Bmatrix} R_{12} & R_3 & R_{123} \\ \varepsilon^{\pm} & R_{1234} & R_{34} \end{Bmatrix} = \tag{57}$$

$$= \sum_{R_{23}} D_{23} \begin{Bmatrix} R_1 & R_2 & R_{12} \\ R_3 & R_{123} & R_{23} \end{Bmatrix} \begin{Bmatrix} R_1 & R_{23} & R_{123} \\ \varepsilon^{\pm} & R_{1234} & R_{234} \end{Bmatrix} \begin{Bmatrix} R_2 & R_3 & R_{23} \\ \varepsilon^{\pm} & R_{234} & R_{34} \end{Bmatrix}$$

We observe that the relation have three primitive 6-j symbols with argument $\varepsilon^{\pm}$.

Although the primitive 6-j symbols general expression is not known, they can be computed in each particular case from the representation theory. We are not interested in them for now, so we treat them as coefficients. Let us focus on the symbols with no $\varepsilon^{\pm}$ in arguments, they form a linear combination that can be seen as a recursion. Indeed, the first row arguments are the same for all non-primitive 6-j symbols, whereas the second row arguments on the left-hand side are $R_{34} \subset R_3 \otimes \varepsilon^{\pm}, R_{1234} \subset R_{123} \otimes \varepsilon^{\pm}, R_{234} \subset R_{23} \otimes \varepsilon^{\pm}$. If one set $\varepsilon^{\pm} = \varepsilon^{+}$, this relation is able to recursively expand each 6-j symbol in terms of primitive ones.

We now want to construct a recursion which does not iterates through R_3 and R_{123}, motivated by the three-term recurrence relation where only the argument which correspond to the order of polynomial change. For this reason we take two recurrence relations from the previous step corresponding to ε^{+} and ε^{-} and make a composition of them. With the standard notation where $R_{ab} \subset R_a \otimes R_b$ two recurrences are:

$$\begin{Bmatrix} R_1 & R_2 & R_{12} \\ R_{345} & R_{12345} & R_{2345} \end{Bmatrix} = \sum_i C_i^{+} \begin{Bmatrix} R_1 & R_2 & R_{12} \\ R_{34} & R_{1234} & R_{234}^{(i)} \end{Bmatrix} \tag{58}$$

$$\begin{Bmatrix} R_1 & R_2 & R_{12} \\ R_{34} & R_{1234} & R_{234}^{(i)} \end{Bmatrix} = \sum_j C_{i,j}^{-} \begin{Bmatrix} R_1 & R_2 & R_{12} \\ R_3 & R_{123} & R_{23}^{(i,j)} \end{Bmatrix} \tag{59}$$

For brevity we denoted all products of primitive 6-j symbols as $C_i^{\pm}$, where $\pm$ sign depends on the choice of $\varepsilon^{\pm}$. In particular, $R_{34} \subset R_3 \otimes \varepsilon^{-}$ and $R_{345} \subset R_{34} \otimes \varepsilon^{+}$.

As the equation is obtained as a specification of the pentagon equation, there is no restrictions on representations except $R_{234}^{(i)}$ and $R_{23}^{(i,j)}$, which are summed up. Hence, we are free to specify for example $R_{345} = R_3$. The worst scenario is that 6-j symbols become zero and the identity is trivial in this case. From the representation theory it is known that in the decomposition of $R \otimes \varepsilon^{-} \otimes \varepsilon^{+}$ there is always representation $R \subset R \otimes \varepsilon^{-} \otimes \varepsilon^{+}$. Thus, we specify $R_{345} = R_3$, $R_{12345} = R_{123}$. After the substitution of one equation into another we obtain the recurrence:

$$\begin{Bmatrix} R_1 & R_2 & R_{12} \\ R_3 & R_{123} & R_{2345} \end{Bmatrix} = \sum_i \sum_j C_i^{+} C_{ij}^{-} \begin{Bmatrix} R_1 & R_2 & R_{12} \\ R_3 & R_{123} & R_{23}^{(i,j)} \end{Bmatrix} \tag{60}$$

Note, that the recurrence is not unique. First, there are in general a freedom in the choice of R_{1234} and R_{34} that leads to a nonequivalent recurrence relation. We also could use the different order in the product $R_3 \otimes \varepsilon^{+} \otimes \varepsilon^{-}$, it leads to a more complex, but equivalent expression. □

Proposition 5. *6-j symbols* $\begin{Bmatrix} J_1 & T_2 & R_{12} \\ J_3 & R_{123} & R_{23} \end{Bmatrix}$ *satisfy a three-term relation.*

Proof. Let us make the recurrence arising from the Lemma 2 the three-term one. We make a specification $R_1 = J_1$, $R_2 = T_2$, $R_3 = J_3$, $R_{34} = J_{34}$. The decomposition of $J_1 \otimes T_2 \otimes J_3$ is multiplicity-free, so the pentagon identity can be written in a multiplicity-free form. The coefficients are expressed in terms of primitive 6-j symbols and quantum dimensions $D_i = \dim(R_{23}^{(i)}), D_{ij} = \dim(R_{23}^{(i,j)})$.

$$C_i^+ C_{ij}^- = \frac{D_i D_{ij} \begin{Bmatrix} J_1 & R_{234}^{(i)} & R_{1234} \\ [\![1]\!] & R_{123} & R_{23}^{(0)} \end{Bmatrix} \begin{Bmatrix} T_2 & J_{34} & R_{234}^{(i)} \\ [\![1]\!] & R_{23}^{(0)} & J_3 \end{Bmatrix} \begin{Bmatrix} J_1 & R_{23}^{(i+j)} & R_{123} \\ \overline{[\![1]\!]} & R_{1234} & R_{234}^{(i)} \end{Bmatrix} \begin{Bmatrix} T_2 & J_3 & R_{23}^{(i+j)} \\ \overline{[\![1]\!]} & R_{234}^{(i)} & J_{34} \end{Bmatrix}}{\begin{Bmatrix} R_{12} & J_{34} & R_{1234} \\ [\![1]\!] & R_{123} & J_3 \end{Bmatrix} \begin{Bmatrix} R_{12} & J_3 & R_{123} \\ \overline{[\![1]\!]} & R_{1234} & J_{34} \end{Bmatrix}} \tag{61}$$

From the Lemma 1 the double sum have 4 terms. We show that the recursion from pentagon equation has only three 6-j distinct symbols, meaning that three out of five 6-j symbols coincide. We want to use the pentagon equation as a recurrence on the non-primitive 6-j symbols. Thus, we assume that all representations in the non-primitive 6-j symbols have no constraints. To show that the double sum is indeed three-term we should use the restrictions on $R_{23}^{(i,j)}$ from primitive 6-j symbols in C_i^+, C_j^-.

All restrictions on $R_{234}^{(i)}$ and $R_{23}^{(i,j)}$ can be found from the following diagram:

$$\begin{array}{cccccc} T_2 & & J_3 & & \varepsilon^- & \varepsilon^+ \\ & R_{23}^{(i,j)} & & J_{34} & & - \\ & & R_{234}^{(i)} & & J_{345} & \\ & & & R_{2345} & & \end{array} \qquad \begin{array}{l} R_{23}^{(i,j)} : \begin{cases} R_{23}^{(i,j)} \otimes \overline{[\![1]\!]} \supset R_{234}^{(i)} \\ T_2 \otimes J_3 \supset R_{23}^{(i,j)} \end{cases}, \\ R_{234}^{(i)} : \begin{cases} R_{234}^{(i)} \otimes [\![1]\!] \supset R_{2345} \\ T_2 \otimes J_{34} \supset R_{234}^{(i)} \end{cases} \end{array} \tag{62}$$

The restriction in the second rows for $R_{234}^{(i)}$ and $R_{23}^{(i,j)}$ is already imposed by non-primitive 6-j symbols and in general leaves more than two terms. The restriction that makes each sum two-term is in the first line for both $R_{234}^{(i)}$ and $R_{23}^{(i,j)}$. We are able to apply Lemma 1 for $R_{234}^{(i)}$, so there are only two non-trivial $R_{234}^{(i)}$. We set $i = -1$ for one of them and $i = -1$ for another.

Example 1. Let $N \geq 4$, T_2, J_i are symmetric representations $R_{2345} = [\![a, b]\!]$, $R_5 = [\![1]\!]$. All possible $R_{234}^{(i)}$'s can be obtained as:

$$R_{2345} \otimes [\![1^{N-1}]\!] = [\![a-1, b]\!] \oplus [\![a, b-1]\!] \oplus \underline{[\![a+1, b+1, 1^{N-3}]\!]} \tag{63}$$

The first summand is absent if $a = b$, so in principle the recurrence can have even lesser terms.

The appropriate $R_{23}^{(i,j)}$ are found in the same way. There are again two-terms and the double sum have four summands, but two of them coincide. This fact can be easily understood from the following example:

Example 2. With assumptions of the previous example there are four appropriate $R_{23}^{(i,j)}$'s that can occur in the sum. We take only appropriate terms from the previous example and multiply them by $[\![1^{N-1}]\!]$:

$$R_{23}^{(i,j)} \subset \left(\bigoplus_i R_{23}^{(i)} \right) \otimes [\![1]\!] = [\![a-1, b+1]\!] \oplus 2[\![a, b]\!] \oplus [\![a+1, b-1]\!] \oplus \underline{[\![a-1, b, 1]\!]} \oplus \underline{[\![a, b-1, 1]\!]} \tag{64}$$

If $0 \leq a - b < 2$ some of these terms are zero. The same argument can be applied to the other classes of representations J_3, namely conjugate to symmetric and antisymmetric.

We set $j = \pm 1$ in a way that two coinciding terms have (i, j) equal to $(1. -1), (-1, 1)$. Then we improve our notation and denote $R_{23}^{(i,j)}$ by $R_{23}^{(i+j)}$, we have just shown it is consistent. We specify $R_{2345} = R_{23}^{(0)}$, which makes the recurrence three-term. The resulting relation has the form:

$$\sum_{i=\pm1}\sum_{j=\pm1}\left(D_iD_{ij}\begin{Bmatrix} J_1 & R_{234}^{(i)} & R_{1234} \\ [\![1]\!] & R_{123} & R_{23}^{(0)} \end{Bmatrix}\begin{Bmatrix} T_2 & J_{34} & R_{234}^{(i)} \\ [\![1]\!] & R_{23}^{(0)} & J_3 \end{Bmatrix}\times\right.$$
$$\left.\times\begin{Bmatrix} J_1 & R_{23}^{(i+j)} & R_{123} \\ \overline{[\![1]\!]} & R_{1234} & R_{234}^{(i)} \end{Bmatrix}\begin{Bmatrix} T_2 & J_3 & R_{23}^{(i+j)} \\ \overline{[\![1]\!]} & R_{234}^{(i)} & J_{34} \end{Bmatrix}\times\begin{Bmatrix} J_1 & T_2 & R_{12} \\ J_3 & R_{123} & R_{23}^{(i+j)} \end{Bmatrix}\right)= \tag{65}$$
$$=\begin{Bmatrix} R_{12} & J_{34} & R_{1234} \\ [\![1]\!] & R_{123} & J_3 \end{Bmatrix}\begin{Bmatrix} R_{12} & J_3 & R_{123} \\ \overline{[\![1]\!]} & R_{1234} & J_{34} \end{Bmatrix}\times\begin{Bmatrix} J_1 & T_2 & R_{12} \\ J_3 & R_{123} & R_{23}^{(0)} \end{Bmatrix}$$

□

Now let us identify the polynomial which correspond to 6-j symbol.

The three-term recurrence relation for orthogonal polynomial $y_n(x,\alpha)$ of order $n=0\dots L$ in variable $\nu(x)$, $x=0\dots L$ with parameters $\alpha=\{\alpha_i\}$ has the form:

$$A_ny_{n+1}(x,\alpha)+B_ny_n(x,\alpha)+C_ny_{n-1}(x,\alpha)=\nu(x)y_n(x,\alpha) \tag{66}$$

We assume that coefficients depend on α: $A_n(\alpha),B_n(\alpha),C_n(\alpha)$, but for brevity we omit it. As we can see, the only changing argument is degree n. In the case of 6-j symbols the recurrence is connected with representation R_{23}. For the product $T_2\otimes J_3$ one can introduce a linear ordering of $R_{23}^{(2j)}$ by parameter j. In our notation parameter j takes even integer values and it is bounded by some $j_{\min}$ and $j_{\max}$. Their values can be understood from the fusion rules, because there are only finite non-trivial $R_{23}^{(2j)}$.

Example 3. Let us consider symmetric $J_i=[\![s_i]\!]$, $T_2=[\![s_2^k]\!]$ and $s_2\geq s_3$. Then the 6-j symbol we write the recurrence for is

$$\begin{Bmatrix} [\![s_1]\!] & [\![s_2^k]\!] & R_{12} \\ [\![s_3]\!] & R_{123} & R_{23}^{(2j)} \end{Bmatrix},\qquad R_{23}^{2j}=[\![s_2+s_3-j,s_2^{k-1},j]\!]=[\![s_2+\widetilde{j},s_2^{k-1},s_3-\widetilde{j}]\!] \tag{67}$$

Two possible choices of j are presented. To connect this with polynomial we need to identify i with n, one can chose any of two possibilities, it is just a convention to work either with $y_n(x,\alpha)$ and $\nu(x)$ or $y_{-n}(x,\alpha)$ and $\nu(x)^{-1}$.

Thus, we state that $n=j+\text{const}$ and $L=|j_{\max}-j_{\min}|$. The same situation with x, which is identified with R_{12}. We define $x=0\dots L$ and $x=i+\text{const}$ in $R_{12}^{(i)}\subset J_1\otimes T_2$.

The value of $\nu(x)$ is just a factor in front of $y_n(x,\alpha)$ that should depend on x. From the perspective of 6-j symbols it is

$$\nu(x)=\left|\begin{Bmatrix} R_{12}^{(x)} & J_{34} & R_{1234} \\ [\![1]\!] & R_{123} & J_3 \end{Bmatrix}\right|^2 \tag{68}$$

It is the only term that has argument R_{12}, so we use it as a definition of variable $\nu(x)$. All remaining terms in front of $y_n(x,\alpha)$ are summands in B_n, that is, the terms with $i+j=0$ in the sum. A_n and C_n are determined analogously, A_n is in front of the $y_{n+1}(x,\alpha)$, hence it is coming from he term with $i+j=2$.

As the three-term relation is homogeneous we can multiply all polynomials by the same $f(\alpha)$ and still get a solution, so it is not unique. To fix the dependence on parameters we need orthogonality relation. It fixes all remaining freedom and the polynomial is uniquely determined.

Let us summarize the result.

$$A_n \begin{Bmatrix} J_1 & T_2 & R_{12} \\ J_3 & R_{123} & R_{23}^{(-2)} \end{Bmatrix} + B_n \begin{Bmatrix} J_1 & T_2 & R_{12} \\ J_3 & R_{123} & R_{23}^{(0)} \end{Bmatrix} + C_n \begin{Bmatrix} J_1 & T_2 & R_{12} \\ J_3 & R_{123} & R_{23}^{(+2)} \end{Bmatrix} = \nu(x) \begin{Bmatrix} J_1 & T_2 & R_{12} \\ J_3 & R_{123} & R_{23}^{(0)} \end{Bmatrix} \tag{69}$$

$$y_n(x) = \begin{Bmatrix} J_1 & T_2 & R_{12} \\ J_3 & R_{123} & R_{23}^{(0)} \end{Bmatrix}, \qquad \nu(x) = \left| \begin{Bmatrix} R_{12} & J_{34} & R_{1234} \\ [\![1]\!] & R_{123} & J_3 \end{Bmatrix} \right|^2 \tag{70}$$

$$B_n = \sum_{i=\pm 1} D_{234}^{(i)} D_{23}^{(0)} \left| \begin{Bmatrix} J_1 & R_{234}^{(i)} & R_{1234} \\ [\![1]\!] & R_{123} & R_{23}^{(0)} \end{Bmatrix} \begin{Bmatrix} T_2 & J_{34} & R_{234}^{(i)} \\ [\![1]\!] & R_{23}^{(0)} & J_3 \end{Bmatrix} \right|^2 \tag{71}$$

$$\left.\begin{matrix} i=1 & A_n \\ i=-1 & C_n \end{matrix}\right] = D_{234}^{(i)} D_{23}^{(2i)} \begin{Bmatrix} J_1 & R_{234}^{(i)} & R_{1234} \\ [\![1]\!] & R_{123} & R_{23}^{(0)} \end{Bmatrix} \begin{Bmatrix} T_2 & J_{34} & R_{234}^{(i)} \\ [\![1]\!] & R_{23}^{(0)} & J_3 \end{Bmatrix} \times \\ \times \begin{Bmatrix} J_1 & R_{23}^{(2i)} & R_{123} \\ \overline{[\![1]\!]} & R_{1234} & R_{234}^{(i)} \end{Bmatrix} \begin{Bmatrix} T_2 & J_3 & R_{23}^{(2i)} \\ \overline{[\![1]\!]} & R_{234}^{(i)} & J_{34} \end{Bmatrix} \tag{72}$$

2.2 Three-term relation for general multiplicity-free 6-j symbols

The method applied in this section can be applied in a far more general setting. In fact, we are able to do exactly the same procedure with an arbitrary multiplicity-free 6-j symbol and obtain a three-term relation.

By the words "general multiplicity-free 6-j symbol" we wean the 6-j symbol of the form $\begin{Bmatrix} J_1 & R_2 & R_{12} \\ J_3 & R_{123} & R_{23} \end{Bmatrix}$ or any other 6-j symbol obtained from it by tetrahedral symmetries. It is general in a sense that if R_i can be arbitrary the only J_1, J_3 that leave the 6-j symbol multiplicity-free are either symmetric, or conjugate to symmetric or antisymmetric representations. There are many other multiplicity-free 6-j symbols, but they have restrictions on the form of at least three arguments, whereas we are considering the 6-j symbol with only two specifications.

All representations are defined as in the previous subsection. J_{34} is uniquely determined as the only $J_{34} \subset J_3 \otimes \varepsilon^-$. Considering R_{1234}, we need to specify it as it can not be determined from fusion rules. We denote $e_i = (0, \dots, 0, 1, 0, \dots, 0)$, $1 \le i \le N$, so R_{1234} can always be expressed through $R_{123} = [\![\mu_1, \dots, \mu_N]\!]$ as $R_{1234} = R_{123}^{(-e_i)} = [\![\mu_1, \dots, \mu_{i-1}, \mu_i - 1, \mu_{i+1}, \dots, \mu_N]\!]$.

Analogously, setting $R_{2345} = [\![\mu_1, \dots, \mu_N]\!]$, we can write $R_{234}^{(-e_i)}$ and $R_{23}^{(-e_i+e_j)}$. Note that the summation in the pentagon is carried out over all i, j coming from fusion rules. It is easy to show that the number of possible i and j is the same and we denote it by $b \le N$. For instance, in the previous subsection we had $R_{23} = [\![a, b^k, c]\!]$ and $i, j \in \{1, k+2\}$ from fusion rules, so $b = 2$. The equality between the number of terms for e_i and e_j can be derived similar to the Lemma 1 but with greater number of terms in the fusions. Moreover, analogously one can show that there are b distinct R_{1234} that do not break fusion rules.

However, in the general situation 6-j symbol can have have $b = N$, so the relation from pentagon equation will have up to $N^2 + 1$ terms, they differ only by $R_{23}^{(-e_i+e_j)}$, with up to $N(N-1)+1$ distinct non-primitive 6-j symbols. This relation is by no means three-term, so there is no corresponding orthogonal polynomial of one variable. But it is reasonable to suppose that 6-j symbols are orthogonal polynomials in several variables. Let us recall how the main statements are generalized for multiple variables [20].

General properties of orthogonal polynomials in several variables. We use the standard multi-index notation for monomials of d variables $x = (x_1, \dots, x_d)$ and powers $x^\alpha = x_1^{\alpha_1} \dots x_d^{\alpha_d}$ for $d \geq 1$. The number $|\alpha| = \sum_i \alpha_i$ is called the total degree. A polynomial P has the form

$$P(x) = \sum_\alpha c_\alpha x^\alpha, \qquad c_\alpha \in \mathbb{C} \tag{73}$$

The main problem difference between orthogonal polynomials in one variable and orthogonal polynomials in several variables is the ordering. Monomials of one variable can be easily ordered by their degree, $\deg(x^{\alpha_1}) < \deg(x^{\alpha'_1})$ if $\alpha < \alpha'$. If we have several variables, the relation $\alpha < \alpha'$ can not be naturally defined. There is a natural grading by the total degree, but inside the subspace of $|\alpha| = n$ there are several possible orderings, e.g. lexicographical.

Due to this difficulty, it is convenient to consider homogeneous polynomials, in which each polynomial has the same total degree. The space $\widetilde{\Pi}_n^d$ of such polynomials is graded:

$$\widetilde{\Pi}_n^d = \{P : P(x) = \sum_{|\alpha|=n} c_\alpha x^\alpha\}, \qquad r_n^d := \dim \widetilde{\Pi}_n^d = \binom{n+d-1}{n} \tag{74}$$

As arbitrary polynomial of total degree n can be expressed in terms of the homogeneous polynomials by setting $x_d = 1$. The space of arbitrary polynomials P_n^d in d variables with monomials of total degree n or lower is denoted by Π_n^d. We deduce that

$$\Pi_n^d = \{P : P(x) = \sum_{|\alpha|\leq n} c_\alpha x^\alpha\}, \qquad \dim \Pi = \binom{n+d}{n} \tag{75}$$

Definition 13. We define the orthogonality with respect to a bilinear form $\langle \bullet, \bullet \rangle$ on Π^d. Given a polynomial P, it is said to be orthogonal polynomial if for all orthogonal polynomials of lower total degree Q it satisfies

$$\langle P_\alpha^n, Q_{\alpha'}^m \rangle = 0, \qquad \forall Q \in \Pi^d, \qquad \text{if } m < n \tag{76}$$

It is convenient to write a set of orthogonal polynomials of total degree n as a vector of, say, lexicographically ordered polynomials

$$\vec{P}_n = (P_\alpha^n)_{|\alpha|=n} = (P_{\alpha^{(1)}}^n, \dots, P_{\alpha^{(r_n^d)}}^n,)^T \tag{77}$$

Theorem 2. *([20]) For $n \geq 0$, $1 \leq i \leq d$ there exist unique matrices $\boldsymbol{A}_{n,i} : r_n^d \times r_{n+1}^d$, $\boldsymbol{B}_{n,i} : r_n^d \times r_n^d$ and $\boldsymbol{C}_{n,i} : r_n^d \times r_{n-1}^d$ such that*

$$x_i \vec{P}_n = \boldsymbol{A}_{n,i} \vec{P}_{n+1} + \boldsymbol{B}_{n,i} \vec{P}_n + \boldsymbol{C}_{n,i} \vec{P}_{n-1}, \qquad 1 \leq i \leq d \tag{78}$$

where we define $\vec{P}_{-1} = 0$ and $\boldsymbol{C}_{-1,i} = 0$.

For a given polynomial P_α^n there are precisely d distinct three-term relations, each iterates on degree n and produces monomial x_i. In each relation the number of terms is up to $1 + r_{n-1}^d + r_n^d + r_{n+1}^d$, so in general orthogonal polynomials may have a cumbersome generalized three-term recurrence relation.

The Favard's theorem can also be generalized in the following way.

Theorem 3. *Let $\{\vec{P}_n\}_{n=0}^\infty = \{P_\alpha^n : |\alpha| = n, n \in \mathbb{N}_0\}$, $\vec{P}_0 = 1$, be an arbitrary sequence in Π^d. Then the following statements are equivalent.*

- *There exists a linear functional $\mathcal{L}$ which defines a quasi-definite linear functional on Π^d and which makes $\{P_n\}_{n=0}^{\infty}$ an orthogonal basis in Π^d.*
- *For $n \geq 0$, $1 \leq i \leq d$, there exist matrices $\boldsymbol{A}_{n,i}, \boldsymbol{B}_{n,i}, \boldsymbol{C}_{n,i}$ such that*
  - *the polynomials $\vec{P}_n$ satisfy the generalized three-term relation (78),*
  - *the matrices in the relation satisfy the rank conditions:*

$$\operatorname{rk} \boldsymbol{A}_{n,i} = \operatorname{rk} \boldsymbol{C}_{n+1,i} = r_n^d \tag{79}$$

Three-term relation for 6-j symbols.

Proposition 6. *6-j symbols $\begin{Bmatrix} J_1 & R_2 & R_{12} \\ J_3 & R_{123} & R_{23} \end{Bmatrix}$ satisfy a generalized three-term relation and form orthogonal polynomials.*

The relation obtained from pentagon identity has the form:

$$\begin{gathered}\sum_i \sum_j \left(D_i D_{ij} \begin{Bmatrix} J_1 & R_{234}^{(\alpha-e_i)} & R_{1234}^{(e_k)} \\ [\![1]\!] & R_{123} & R_{23}^{(\alpha)} \end{Bmatrix} \begin{Bmatrix} R_2 & J_{34} & R_{234}^{(\alpha-e_i)} \\ [\![1]\!] & R_{23}^{(\alpha)} & J_3 \end{Bmatrix} \times \right. \\ \times \begin{Bmatrix} J_1 & R_{23}^{(\alpha-e_i+e_j)} & R_{123} \\ \overline{[\![1]\!]} & R_{1234}^{(e_k)} & R_{234}^{(\alpha-e_i)} \end{Bmatrix} \begin{Bmatrix} R_2 & J_3 & R_{23}^{(\alpha-e_i+e_j)} \\ \overline{[\![1]\!]} & R_{234}^{(\alpha-e_i)} & J_{34} \end{Bmatrix} \times \left. \begin{Bmatrix} J_1 & R_2 & R_{12} \\ J_3 & R_{123} & R_{23}^{(\alpha-e_i+e_j)} \end{Bmatrix} \right) = \\ = \begin{Bmatrix} R_{12} & J_{34} & R_{1234}^{(e_k)} \\ [\![1]\!] & R_{123} & J_3 \end{Bmatrix} \begin{Bmatrix} R_{12} & J_3 & R_{123} \\ \overline{[\![1]\!]} & R_{1234}^{(e_k)} & J_{34} \end{Bmatrix} \times \begin{Bmatrix} J_1 & R_2 & R_{12} \\ J_3 & R_{123} & R_{23}^{(\alpha)} \end{Bmatrix}\end{gathered} \tag{80}$$

To interpret this relation as a generalized three-term one, we introduce variables:

$$x_k := \begin{Bmatrix} R_{12} & J_{34} & R_{1234}^{(e_k)} \\ [\![1]\!] & R_{123} & J_3 \end{Bmatrix} \begin{Bmatrix} R_{12} & J_3 & R_{123} \\ \overline{[\![1]\!]} & R_{1234}^{(e_k)} & J_{34} \end{Bmatrix} \tag{81}$$

From the discussion in the beginning of this subsection we have found that there are b nonequivalent recurrence relations, enumerated by $1 \leq k \leq b$. It can be seen that x_k does not depend on R_{23} and R_{12}, which we associate with degrees and coordinates.

There are several ways to introduce the ordering in 6-j symbols by degree, we use the following one. For $R_{23} \in R_2 \otimes J_3$ we write a vector $\alpha = (\alpha_1, \dots, \alpha_b)$ encoding the number of boxes added to the Young diagram. Obviously, $\sum_i \alpha_i = |J_3|$. We set $n = |J_3| - \alpha_1$, let us examine it. Indeed, if $n = 0$, then R_{23} is uniquely fixed. If $n = 1$, there are b polynomials and so on.

We group the recurrence relation according to this ordering.

$$\begin{gathered}\sum_{i=2}^{b} A_{\alpha i} \begin{Bmatrix} J_1 & R_2 & R_{12} \\ J_3 & R_{123} & R_{23}^{(\alpha-e_1+e_i)} \end{Bmatrix} + \sum_{i=2}^{b} C_{\alpha i} \begin{Bmatrix} J_1 & R_2 & R_{12} \\ J_3 & R_{123} & R_{23}^{(\alpha+e_1-e_i)} \end{Bmatrix} = \\ = \sum_{i=2}^{b} \sum_{j=2}^{b} B_{\alpha ij} \begin{Bmatrix} J_1 & R_2 & R_{12} \\ J_3 & R_{123} & R_{23}^{(\alpha+e_i-e_j)} \end{Bmatrix} + (B_{\alpha 11} + x_i) \begin{Bmatrix} J_1 & R_2 & R_{12} \\ J_3 & R_{123} & R_{23}^{(\alpha)} \end{Bmatrix}\end{gathered} \tag{82}$$

Now the identification between generalized three-term relation and the relation above is obvious.

The statement above holds for arbitrary q. There is a result for $q = 1$ known for a long time [21], that 6-j symbols indeed form an orthogonal polynomial. The three-term relation is obtained explicitly for the known expression of multiplicity-free 6-j symbol.

3 Pentagon solution for types I^+ and II^+

In this section we consider 6-j symbols of types I^+ and II^+. They are multiplicity-free and iterable, so they can be written as some orthogonal polynomials. We are interested in these 6-j symbols because the three-term relation for them can be written explicitly. The corresponding polynomials are tightly connected with the Racah polynomials and we can express a general 6-j symbol of types I^+ and II^+ in terms of some Racah polynomial. This statement can be formulated more precisely in the following way.

Proposition 7. *All 6-j symbols of type I^+ and II^+ can be written as some Saalschützian ${}_4\Phi_3$:*

$$\begin{bmatrix} r_1 & r_2 & i \\ r_3 & r_4 & j \end{bmatrix}_T^N = K_T \cdot {}_4\Phi_3 \left(\begin{matrix} a_1, a_2, a_3, a_4 \\ b_1, b_2, b_3 \end{matrix} ; q, q \right), \tag{83}$$

$$2a_i = \begin{pmatrix} -s_1 - s_2 + r_{12} - 2(N-2)\delta_{T,2} \\ -s_3 - s_{123} + r_{12} \\ -s_1 - s_{123} + r_{23} \\ -s_2 - s_3 + r_{23} \end{pmatrix}, \qquad 2b_i = \begin{pmatrix} -s_1 - s_2 - s_3 - s_{123} - 2(N-1) \\ r_{12} + r_{23} - s_2 - s_{123} + 2 \\ r_{12} + r_{23} - s_1 - s_3 + 2 + 2(N-2)\delta_{T,1} \end{pmatrix}. \tag{84}$$

with some monomial coefficient K_T that depends on type and representations.

The calculation of coefficients is possible due to the following result [8], which is discussed in detail in Appendix A:

Proposition 8. *Any multiplicity-free Racah matrix U of size 2 by 2 is of the form:*

$$U = \begin{pmatrix} -u_{11} & u_{12} \\ u_{12} & u_{11} \end{pmatrix}$$
$$u_{11} = \sqrt{\frac{[\kappa_{12} - \kappa_{13} - \kappa_{23}]\,[\kappa_{23} - \kappa_{12} - \kappa_{13}]}{[2\kappa_{12}]\,[2\kappa_{23}]}}, \qquad u_{12} = \sqrt{\frac{[\kappa_{12} + \kappa_{23} - \kappa_{13}]\,[\kappa_{12} + \kappa_{23} + \kappa_{13}]}{[2\kappa_{12}]\,[2\kappa_{23}]}} \tag{85}$$

where $q^{\kappa_{12}}, q^{\kappa_{23}}, q^{\kappa_{13}}$ are the eigenvalues of normalized matrices $\mathcal{R}_{12}, \mathcal{R}_{23}, \mathcal{R}_{13}$, $\kappa_{ij} \geq 0$, expressed in terms of Casimir eigenvalues:

$$\lambda_i(\mathcal{R}_{12}) = q^{\kappa_{Q_i} - \kappa_{R_1} - \kappa_{R_2}} \quad \textit{for } Q_i \subset R_1 \otimes R_2, \qquad q^{\kappa_{12}} := \frac{\lambda_1}{\sqrt{\lambda_1 \lambda_2}} = q^{\frac{1}{2}(\kappa_{Q_1} - \kappa_{Q_2})} \tag{86}$$

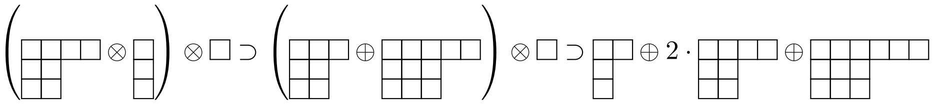
3.1 Orthogonal polynomials for type I^+

In this subsection we derive the expression for type I^+ 6-j symbol. The technique requires a lot of calculation, so we split it into several parts:

1. Write a three-term relation on the 6-j symbol.
2. Express all coefficients as 2x2 Racah matrices.
3. Associate 6-j symbol and Racah polynomial.

Three-term relation. Type I^+ 6-j symbols are iterable. The three-term relation can be constructed as in the general case. An example of the $R_{23} \otimes \overline{[\![1]\!]} \otimes [\![1]\!]$ decomposition for $N = 4$. Representations $R_{23}, R_{234}, R_{2345}$ are restricted to the form $[\![a, b^{N-2}]\!]$ by fusion rules.

Example 4. Let us specify $N=4$ and $R_{23}=[\![4,2^2]\!]$. The decomposition of $R_{23}\otimes\varepsilon^-\otimes\varepsilon^+$ is:



(87)

We specify $R_{23}^{(i)}$ as follows. $R_{23}^{(0)}=R_{2345}=[\![4,2^2]\!]$, $R_{23}^{2i}=[\![4+2i,(2+i)^2]\!]$ for $i\in\{-1,0,1\}$. Representation $R_{234}^{(i)}$ corresponds to $[\![3+i,(2+\delta_{i,1})^2]\!]$. Indices are representing the width of a Young diagram.

For a general case we denote:

$$\begin{aligned} R_{12}&=\left[\!\!\left[r_{12},\frac{r_{12}+s_2-s_1}{2}^{N-2}\right]\!\!\right], \\ R_{23}^{(2i)}&=\left[\!\!\left[r_{23}+2i,\frac{r_{23}+2i+s_2-s_3}{2}^{N-2}\right]\!\!\right], \qquad i\in\{-1,0,1\}, \\ R_{234}^{(i)}&=\left[\!\!\left[r_{23}+i,\frac{r_{23}+2\delta_{i,1}+s_2-s_3}{2}^{N-2}\right]\!\!\right], \qquad i\in\{+1,-1\}. \end{aligned} \tag{88}$$

The three-term relation for type I$^+$ is:

$$\sum_{i=\pm1}\sum_{j=\pm1}\left(D_{234}^{(i)}D_{23}^{(2i)}\begin{Bmatrix} S_1 & R_{234}^{(i)} & S_{1234} \\ [\![1]\!] & R_{123} & R_{2345}\end{Bmatrix}\begin{Bmatrix} \overline{S}_2 & S_{34} & R_{234}^{(i)} \\ [\![1]\!] & R_{23}^{(0)} & S_3\end{Bmatrix}\times\right.$$
$$\left.\times\begin{Bmatrix} S_1 & R_{23}^{(i+j)} & R_{123} \\ \overline{[\![1]\!]} & S_{1234} & R_{234}^{(i)}\end{Bmatrix}\begin{Bmatrix} \overline{S}_2 & S_3 & R_{23}^{(i+j)} \\ \overline{[\![1]\!]} & R_{234}^{(i)} & S_{34}\end{Bmatrix}\times\begin{bmatrix} s_1 & s_2 & r_{12} \\ s_3 & s_{123} & r_{23}+i+j\end{bmatrix}_1\right)=$$
$$=\begin{Bmatrix} R_{12} & S_{34} & S_{1234} \\ [\![1]\!] & R_{123} & S_3\end{Bmatrix}\begin{Bmatrix} R_{12} & S_3 & R_{123} \\ \overline{[\![1]\!]} & S_{1234} & S_{34}\end{Bmatrix}\times\begin{bmatrix} s_1 & s_2 & r_{12} \\ s_3 & s_{123} & r_{23}\end{bmatrix}_1$$

Or just

$$\begin{aligned} A(r_{23})\begin{bmatrix} s_1 & s_2 & r_{12} \\ s_3 & s_{123} & r_{23}+2\end{bmatrix}_1 + B(r_{23})\begin{bmatrix} s_1 & s_2 & r_{12} \\ s_3 & s_{123} & r_{23}\end{bmatrix}_1 + C(r_{23})\begin{bmatrix} s_1 & s_2 & r_{12} \\ s_3 & s_{123} & r_{23}-2\end{bmatrix}_1 &= \\ =\nu(r_{12})\begin{bmatrix} s_1 & s_2 & r_{12} \\ s_3 & s_{123} & r_{23}\end{bmatrix}_1 \end{aligned} \tag{89}$$

The coefficients in the three-term relation are:

$$\begin{aligned} \nu(r_{12}) &= \left|\begin{Bmatrix} R_{12} & S_3 & R_{123} \\ \overline{[\![1]\!]} & S_{1234} & S_{34}\end{Bmatrix}\right|^2 \\ \begin{matrix} i=1 & A(r_{23}) \\ i=-1 & C(r_{23})\end{matrix}\Bigg] &= D_{234}^{(2n+i)}D_{23}^{(2n+2i)}\begin{Bmatrix} S_1 & R_{234}^{(i)} & R_{1234} \\ [\![1]\!] & R_{123} & R_{2345}\end{Bmatrix}\begin{Bmatrix} \overline{S}_2 & S_{34} & R_{234}^{(i)} \\ [\![1]\!] & R_{23}^{(0)} & S_3\end{Bmatrix}\times \\ &\times\begin{Bmatrix} S_1 & R_{23}^{(2i)} & R_{123} \\ \overline{[\![1]\!]} & R_{1234} & R_{234}^{(i)}\end{Bmatrix}\begin{Bmatrix} \overline{S}_2 & S_3 & R_{23}^{(2i)} \\ \overline{[\![1]\!]} & R_{234}^{(i)} & S_{34}\end{Bmatrix} \\ B(r_{23}) = \sum_i B_i(r_{23}) &= \sum_i D_{234}^{(2n+i)}D_{23}^{(2n)}\left|\begin{Bmatrix} S_1 & R_{234}^{(i)} & R_{1234} \\ [\![1]\!] & R_{123} & R_{23}^{(0)}\end{Bmatrix}\begin{Bmatrix} \overline{S}_2 & S_{34} & R_{234}^{(i)} \\ [\![1]\!] & R_{23}^{(0)} & S_3\end{Bmatrix}\right|^2 \end{aligned} \tag{90}$$

Primitive 6-j symbols calculation. The primitive 6-j symbols above are elements of 2x2 Racah matrices. The matrices of this kind was found explicitly from the cabling procedure [8], we process this result in the

Appendix A. The obtained 6-j symbols are as follows.

The $\nu(r_{12})$ term:

$$\nu(r_{12}) = \left| \begin{Bmatrix} R_{12} & [\![s_{34}]\!] & R_{1234} \\ [\![1]\!] & R_{123} & [\![s_3]\!] \end{Bmatrix} \right|^2 = \begin{vmatrix} 2\kappa_{12} = s_{123} + N - 1 \\ 2\kappa_{23} = s_3 \\ 2\kappa_{13} = r_{12} + N - 1 \end{vmatrix}_{+}^{-} = \frac{\left[\frac{s_3 - r_{12} + s_{123}}{2}\right]\left[\frac{r_{12} + s_{123} + s_3 + 2N - 2}{2}\right]}{[s_{123} + N - 1][s_3] D_{1234} D_3} \tag{91}$$

The 6-j symbols forming $A(r_{23}), B(r_{23}), C(r_{23})$ terms:

$$\left| \begin{Bmatrix} [\![s_1]\!] & R_{234}^{(i)} & R_{1234} \\ [\![1]\!] & R_{123} & R_{23}^{(i+j)} \end{Bmatrix} \right|^2 = \begin{vmatrix} 2\kappa_{12} = s_{123} + N - 1 \\ 2\kappa_{23} = r_{23} + N - 1 + i \\ 2\kappa_{13} = s_1 + 1 \end{vmatrix}_{j}^{-} = \frac{\left[\frac{r_{23} + N - 1 + i - j(s_1 - s_{123} - N + 2)}{2}\right]\left[\frac{j(s_1 + s_{123} + N) + r_{23} + N - 1 + i}{2}\right]}{[s_{123} + N - 1][r_{23} + N - 1 + i] D_{1234} D_{23}^{(i+j)}} \tag{92}$$

$$\left| \begin{Bmatrix} \overline{[\![s_2]\!]} & [\![s_3]\!] & R_{23}^{(i+j)} \\ \overline{[\![1]\!]} & R_{234}^{(i)} & [\![s_{34}]\!] \end{Bmatrix} \right|^2 = \begin{vmatrix} 2\kappa_{12} = r_{23} + N - 1 + i \\ 2\kappa_{23} = s_3 + N - 1 \\ 2\kappa_{13} = s_2 + 1 \end{vmatrix}_{-}^{j} = \frac{\left[\frac{j(s_3 + N - 1) - s_2 + r_{23} + N - 2 + i}{2}\right]\left[\frac{s_2 + r_{23} + N + i + j(s_3 + N - 1)}{2}\right]}{[r_{23} + N - 1 + i][s_3 + N - 1] D_{34} D_{23}^{(i+j)}} \tag{93}$$

The coefficients $A(r_{23}), B(r_{23}), C(r_{23}), \nu(r_{12})$ can be simultaneously multiplied by the r_{23}-independent factor without change of the solution. The normalization factor $[s_{123} + N - 1][s_3] D_{1234} D_3$ is chosen to simplify $\nu(r_{12})$. The coefficients in the type I^+ three-term relation (89) are then:

$$\nu(r_{12}) = \left[\frac{s_3 + s_{123} - r_{12}}{2}\right] \left[-\frac{s_3 + s_{123} + r_{12}}{2} - N + 1\right] \tag{94}$$

$$\begin{gathered} B_1(r_{23}) = -\frac{\left[\frac{s_1 - s_{123} + 2 + r_{23}}{2}\right]\left[\frac{-s_{123} - s_1 + r_{23}}{2}\right]\left[\frac{-s_3 - s_2 + r_{23}}{2}\right]\left[\frac{r_{23} - s_3 + s_2 - 2 + 2N}{2}\right]}{[r_{23} + N][r_{23} + N - 1]} - \\ B_2(r_{23}) = -\frac{\left[\frac{-s_1 + s_{123} + 2N - 4 + r_{23}}{2}\right]\left[\frac{s_{123} + s_1 + 2N + r_{23} - 2}{2}\right]\left[\frac{s_2 + r_{23} + 2N - 2 + s_3}{2}\right]\left[\frac{r_{23} - s_2 + s_3}{2}\right]}{[r_{23} + N - 2][r_{23} + N - 1]} \end{gathered} \tag{95}$$

$$\begin{aligned} A(r_{23}) &= \frac{\sqrt{\left[\frac{2N - s_1 + s_{123} - 2 + r_{23}}{2}\right]\left[\frac{s_{123} + s_1 + 2N + r_{23}}{2}\right]\left[\frac{2N - 2 - s_2 + r_{23} + s_3}{2}\right]\left[\frac{s_2 + r_{23} + 2N + s_3}{2}\right]\left[\frac{s_1 - s_{123} + 2 + r_{23}}{2}\right]}}{[r_{23} + N][r_{23} + N - 1]} \times \\ &\times \sqrt{\frac{\left[\frac{-s_{123} - s_1 + r_{23}}{2}\right]\left[\frac{-s_3 - s_2 + r_{23}}{2}\right]\left[\frac{r_{23} - s_3 + s_2 - 2 + 2N}{2}\right]}{\left[\frac{r_{23} - s_3 + s_2 + 2}{2}\right]}} \\ C(r_{23}) &= \frac{\sqrt{\left[\frac{s_1 - s_{123} + r_{23}}{2}\right]\left[\frac{-s_{123} - s_1 + r_{23} - 2}{2}\right]\left[\frac{-s_3 - 2 - s_2 + r_{23}}{2}\right]\left[\frac{r_{23} - s_3 + s_2}{2}\right]\left[\frac{-s_1 + s_{123} + 2N - 4 + r_{23}}{2}\right]}}{[r_{23} + N - 2][r_{23} + N - 1]} \times \\ &\times \sqrt{\frac{\left[\frac{s_{123} + s_1 + 2N + r_{23} - 2}{2}\right]\left[\frac{s_2 + r_{23} + 2N - 2 + s_3}{2}\right]\left[\frac{r_{23} - s_2 + s_3}{2}\right]}{\left[\frac{s_3 - s_2 + r_{23} + 2N - 4}{2}\right]}} \end{aligned} \tag{96}$$

Racah polynomial. The Racah polynomials and some other polynomials are identical if and only if they have the same three-term relation and the polynomials are monic. That is, the coefficients $A_n, B_n, C_n, \nu(x)$ have the same expressions and the polynomial's base value $y_0(x) = 1$. If the latter condition is not fulfilled, then the polynomials will differ by a common factor $y_n(x) = \mathfrak{R}_n(x) \cdot y_0(x)$, as it can be seen from the three-term relation itself. We have a problem with the constructive derivation of this factor, so we apply the following trick. Both of this polynomials by Favard's theorem possess a unique orthogonality relation. For both Racah polynomials and 6-j symbols we know the form of this relation including the exact expression for the measure. As far as the three-term relation coincide for both polynomials, we can extract the factor $y_0(x)$ by equating the measures of orthogonality relations.

We can choose a special normalization that makes our three-term relation and Racah polynomial's three-term relation identical. This normalization is made by the following polynomial redefinition:

$$P_{r_{23}}(r_{12}) = \begin{bmatrix} s_1 & s_2 & r_{12} \\ s_3 & s_{123} & r_{23} \end{bmatrix}_1 \cdot K_{3TR}(r_{23}) \tag{97}$$

The three-term equation is homogeneous, so we can multiply all terms by a common factor and the coefficients $B(r_{23})$, $\nu(r_{12})$ for $P_{r_{23}}(r_{12})$ are the same as for 6-j symbol. Only $A(r_{23})$ and $C(r_{23})$ are changed. We choose the The monomial factor is uniquely determined by the property that new coefficient $C(r_{23})$ is equal to $-B_2(r_{23})$.

$$K_{3TR}(r_{23}) = \sqrt{\frac{\left[\frac{s_1-s_{123}+r_{23}}{2}\right]!\left[\frac{r_{23}-s_3+s_2}{2}\right]!}{\left[\frac{-s_1+s_{123}+2N-4+r_{23}}{2}\right]!\left[\frac{s_{123}+s_1+2N+r_{23}-2}{2}\right]!}}\times \tag{98}$$
$$\times\sqrt{\frac{1}{\left[\frac{s_3-s_2+r_{23}+2N-4}{2}\right]!\left[\frac{s_2+r_{23}+2N-2+s_3}{2}\right]!\left[\frac{s_{123}+s_1-r_{23}}{2}\right]!\left[\frac{s_3+s_2-r_{23}}{2}\right]!}}$$

The fact that $A(r_{23})$ transforms into $-B_1(r_{23})$ tells us that the polynomial has Racah polynomial properties.

The new polynomial has the following three-term relation:

$$A(r_{23})P_{r_{23}}(r_{12}) - (A(r_{23})+C(r_{23}))P_{r_{23}}(r_{12}) + C(r_{23})P_{r_{23}}(r_{12}) = \nu(r_{12})P_{r_{23}}(r_{12}) \tag{99}$$

The variable $\nu(r_{12}) = -\left[\frac{s_3+s_{123}-r_{12}}{2}\right]\left[\frac{s_3+s_{123}+r_{12}}{2}+N-1\right]$ encapsulates R_{12} dependence. The three-term relation coefficients A_n, C_n satisfy the simple relation $B_n + A_n + C_n = 0$:

$$\begin{aligned} A_n &= \frac{\left[\frac{s_1-s_{123}+2+r_{23}}{2}\right]\left[\frac{-s_{123}-s_1+r_{23}}{2}\right]\left[\frac{-s_3-s_2+r_{23}}{2}\right]\left[\frac{r_{23}-s_3+s_2-2+2N}{2}\right]}{[r_{23}+N]\,[r_{23}+N-1]} - \\ C_n &= \frac{\left[\frac{-s_1+s_{123}+2N-4+r_{23}}{2}\right]\left[\frac{s_{123}+s_1+2N+r_{23}-2}{2}\right]\left[\frac{s_2+r_{23}+2N-2+s_3}{2}\right]\left[\frac{r_{23}-s_2+s_3}{2}\right]}{[r_{23}+N-2]\,[r_{23}+N-1]} \end{aligned} \tag{100}$$

These coefficients are of the same form as Racah polynomial coefficients (49). The identification is given by:

$$\begin{cases} a = -s_3 - 1 \\ b = -s_2 + 1 - N \\ d = \frac{s_1-s_{123}-s_3+s_2}{2} \\ g = -\frac{s_{123}+s_1+s_3+s_2}{2} - N \\ n = \frac{s3+s2-r_{23}}{2} \\ x = \frac{s_3+s_{123}-r_{12}}{2} \end{cases} \tag{101}$$

Thus, the polynomial $P_{r_{23}}(r_{12})$ is proportional to Racah polynomial:

$$\begin{aligned} R_{23} &= [\![s_2+s_3-2n,(s_2-n)^{N-2}]\!] \qquad 2n = s_2+s_3-r_{23}, \\ R_{12} &= [\![s_1+s_2-2x,(s_2-x)^{N-2}]\!] \qquad 2x = s_1+s_2-r_{12} \end{aligned} \tag{102}$$

$$\mathfrak{R}_n(\nu(x)) = K(x)\cdot P_{s_2+s_3-2n}(s_3+s_{123}-2x) \tag{103}$$

We can express the 6-j symbol up to a monomial factor $K_{ort} = K(x) \cdot K_{3TR}(n)$ as:

$$\begin{bmatrix} s_1 & s_2 & r_{12} \\ s_3 & s_{123} & r_{23} \end{bmatrix}_1 \cdot K_{ort} = {}_4F_3\left(\begin{matrix} \frac{-s_2-s_3+r_{23}}{2}, \frac{-s_2-s_3-r_{23}}{2} - N + 1, \frac{-s_3-s_{123}+r_{12}}{2}, \frac{-s_3-s_{123}-r_{12}}{2} - N + 1 \\ -s_3, -\frac{s_2+s_3+s_{123}-s_1}{2} - N + 2, -\frac{s_1+s_2+s_3+s_{123}}{2} - N + 1 \end{matrix}; q, q\right) \tag{104}$$

The monomial factor $K(x)$ depends on r_{12} and s_1, s_2, s_3, s_{123}, but invariant on r_{23}. The expression of $K(x)$ is not unity because two polynomials above have different orthogonality measure, although they have the same three-term relation. It does not contradict with the Favard's theorem because we do not require $P_{r_{23}}(r_{12})$ to be equal one for $n = 0$. This is hard to do explicitly because we do not know the value of polynomial at $n = 0$ and we can not obtain the normalization.

On the other hand, we can use the orthogonality relations of both Racah and 6-j symbols and use the Favard's theorem to associate the measures. On the one hand, for non-trivial 6-j symbols:

$$\sum_{r_{12}} \begin{bmatrix} s_1 & s_2 & r_{12} \\ s_3 & s_{123} & r_{23} \end{bmatrix}_1 \begin{bmatrix} s_1 & s_2 & r_{12} \\ s_3 & s_{123} & r_{23} \end{bmatrix}_1 D_{12} D_{23} = 1 \tag{105}$$

On the other hand, if we write it as Racah polynomial, there is such K_{ort} that

$$\sum_x \mathfrak{R}_n(\nu(x)) \mathfrak{R}_n(\nu(x)) \cdot \frac{D_{12} D_{23}}{K^2_{ort}(x, n)} = 1 \tag{106}$$

is the orthogonality of Racah polynomial with the canonical measure (50). It provides us with the proportionality coefficient between 6-j symbol and Racah polynomial. This coefficient K_{ort} is monomial, but cumbersome, it is better to obtain it as the identification between equations (50) and (106).

The known empiric formula [1, 6] is

$$\begin{bmatrix} s_1 & s_2 & r_{12} \\ s_3 & s_4 & r_{23} \end{bmatrix}_1 = K \cdot {}_4\Phi_3\left(\begin{matrix} a_1, a_2, a_3, a_4 \\ b_1, b_2, b_3 \end{matrix}; q, q\right), \tag{107}$$

$$2a_i = \begin{pmatrix} -s_1 - s_2 + r_{12} \\ -s_3 - s_4 + r_{12} \\ -s_1 - s_4 + r_{23} \\ -s_2 - s_3 + r_{23} \end{pmatrix}, \qquad 2b_i = \begin{pmatrix} -s_1 - s_2 - s_3 - s_4 - 2(N-1) \\ r_{12} + r_{23} - s_2 - s_4 + 2 \\ r_{12} + r_{23} - s_1 - s_3 + 2(N-1) \end{pmatrix}. \tag{108}$$

It can be transformed into the same hypergeometric function via Sears' transformation (35):

$${}_4\Phi_3\left(\begin{matrix} x, y, z, n \\ u, v, w \end{matrix}; q, q\right) = \frac{[v-z-n-1]_q![u-z-n-1]_q![v-1]_q![u-1]_q!}{[v-z-1]_q![v-n-1]_q![u-z-1]_q![u-n-1]_q!} \, {}_4\Phi_3\left(\begin{matrix} w-x, w-y, z, n \\ 1-u+z+n, 1-v+z+n, w \end{matrix}; q, q\right), \tag{109}$$

The arguments then become:

$$2a_i = \begin{pmatrix} -s_1 - s_2 + r_{12} \\ -s_1 - s_2 - r_{12} - 2N - 2 \\ -s_1 - s_4 + r_{23} \\ -s_1 - s_4 - r_{23} - 2N + 2 \end{pmatrix}, \qquad 2b_i = \begin{pmatrix} -s_1 - s_2 - s_3 - s_4 - 2(N-1) \\ -2s_1 \\ -2s_1 - 2(N-2) \end{pmatrix}. \tag{110}$$

Which is exactly what we need.

The question about phases in tetrahedral symmetries and the three-term relation can be solved as follows.

The sign can always be chosen in such a way that cyclic permutations do not change the sign, according to Butler [15]. It ensures that we know the signs correctly.

3.2 Orthogonal polynomials for type II$^+$

The $\nu(r_{12})$ term:

$$\nu(r_{12}) = \left|\begin{Bmatrix} R_{12} & \overline{[\![s_{34}]\!]} & R_{1234} \\ [\![1]\!] & R_{123} & \overline{[\![s_3]\!]} \end{Bmatrix}\right|^2 = \begin{vmatrix} 2\kappa_{12} = s_{123} + N - 1 \\ 2\kappa_{23} = s_3 + N \\ 2\kappa_{13} = r_{12} + 1 \end{vmatrix}_{-}^{-} = \frac{[\frac{s_3+r_{12}-s_{123}+2}{2}][\frac{r_{12}+s_{123}-s_3}{2}]}{[s_{123}+N-1][s_3+N]D_{1234}D_3} \tag{111}$$

The 6-j symbols forming $A(r_{23}), B(r_{23}), C(r_{23})$ terms:

$$\left|\begin{Bmatrix} [\![s_1]\!] & R_{234}^{(i)} & R_{1234} \\ [\![1]\!] & R_{123} & R_{23}^{(i+j)} \end{Bmatrix}\right|^2 = \begin{vmatrix} 2\kappa_{12} = s_{123} + N - 1 \\ 2\kappa_{23} = r_{23} + N - 1 + i \\ 2\kappa_{13} = s_1 + 1 \end{vmatrix}_{j}^{-} = \frac{[\frac{r_{23}+N-1+i-j(s_1-s_{123}-N+2)}{2}][\frac{j(s_1+s_{123}+N)+r_{23}+N-1+i}{2}]}{[s_{123}+N-1][r_{23}+N-1+i]D_{1234}D_{23}^{(i+j)}} \tag{112}$$

$$\left|\begin{Bmatrix} [\![s_2]\!] & \overline{[\![s_3]\!]} & R_{23}^{(i+j)} \\ \overline{[\![1]\!]} & R_{234}^{(i)} & \overline{[\![s_{34}]\!]} \end{Bmatrix}\right|^2 = \begin{vmatrix} 2\kappa_{12} = r_{23} + N - 1 + i \\ 2\kappa_{23} = s_3 + 1 \\ 2\kappa_{13} = s_2 + N - 1 \end{vmatrix}_{+}^{j} = \frac{[\frac{-j(s_3+1)-s_2+r_{23}+i}{2}][\frac{s_2+r_{23}+2N-2+i-j(s_3+1)}{2}]}{[r_{23}+N-1+i][s_3+1]D_{34}D_{23}^{(i+j)}} \tag{113}$$

Three-term relation has coefficients:

$$\nu(r_{12}) = \left[\frac{s_3 + r_{12} - s_{123} + 2}{2}\right]\left[\frac{r_{12} + s_{123} - s_3}{2}\right] \tag{114}$$

$$\begin{aligned} A(r_{23}) &= \frac{\left[\frac{s_1-s_{123}+2+r23}{2}\right]\left[\frac{-s_{123}-s_1+r_{23}}{2}\right]\left[\frac{s_3+2N+s_2+r_{23}}{2}\right]\left[\frac{r_{23}-s_2+s_3-2+2N}{2}\right]}{[r_{23}+N-1]\,[r_{23}+N]} \\ C(r_{23}) &= \frac{\left[\frac{-s_1+s_{123}+2N-4+r_{23}}{2}\right]\left[\frac{s_{123}+s_1+2N+r_{23}-2}{2}\right]\left[\frac{-s_3-2-s_2+r_{23}}{2}\right]\left[\frac{r_{23}-s_3+s_2}{2}\right]}{[r_{23}+N-2]\,[r_{23}+N-1]} \end{aligned} \tag{115}$$

It is Racah polynomial with parameters:

$$\begin{cases} \alpha = -s_{123} + 1 - N \\ \beta = -s_1 - 1 \\ \delta = \frac{s_1-s_{123}-s_2+s_3}{2} \\ \gamma = \frac{s_3+s_2-s_1-s_{123}}{2} \\ n = \frac{s_{123}+s_1-r_{23}}{2} \\ x = \frac{r_{12}+s_{123}-s_3}{2} \end{cases} \tag{116}$$

The ${}_4\Phi_3$ function is of the form:

$${}_4\Phi_3\left(\begin{matrix} \frac{-s_{123}-s_1+r_{23}}{2}, \frac{-s_{123}-s_1-r_{23}}{2} - N + 1, \frac{-r_{12}-s_{123}+s_3}{2}, \frac{r_{12}-s_{123}+s_3}{2} + 1 \\ -s_{123} - N + 2, \frac{-s_2+s_3-s_1-s_{123}}{2}, \frac{s_3+s_2-s_1-s_{123}}{2} + 1 \end{matrix}; q, q\right) \tag{117}$$

It can be transformed via two Sears' transformations to

$$
{}_4\Phi_3\left(\begin{matrix}\frac{-s_{123}-s_1+r_{23}}{2}, \frac{-s_{123}-s_1-r_{23}}{2}+1-N, \frac{r_{12}-s_2-s_1}{2}, \frac{-r_{12}-s_2-s_1}{2}-1 \\ -s_1, \frac{-s_2+s_3-s_1-s_{123}}{2}, \frac{-s_3-s_2-s_1-s_{123}}{2}-N+1\end{matrix}; q, q\right) \tag{118}
$$

and then to

$$
{}_4\Phi_3\left(\begin{matrix}\frac{-s_{123}-s_1+r_{23}}{2}, \frac{-s_3-s_2+r_{23}}{2}, \frac{r_{12}-s_2-s_1}{2}, \frac{r_{12}-s_{123}-s_3+2-N}{2} \\ \frac{r_{23}-s_2-s_{123}+r_{12}}{2}+1, 1+\frac{-s_3-s_1+r_{23}+r_{12}}{2}, \frac{-s_3-s_2-s_1-s_{123}}{2}-N+1\end{matrix}; q, q\right) \tag{119}
$$

As is can be seen from the arguments, the hypergeometric function coincide with one written in the Proposition 7, this completes the proof.

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References

[1] Satoshi Nawata, P. Ramadevi, and Zodinmawia. Multiplicity-free quantum 6j-symbols for $U_q(sl_N)$. *Lett. Math. Phys.*, 103:1389–1398, 2013. arXiv:1302.5143, doi:10.1007/s11005-013-0651-4.

[2] C.R. Lienert and P.H. Butler. Racah-wigner algebra for q-deformed algebras. *Journal of Physics A: Mathematical and General*, 25(5):1223, 1992.

[3] A. Klimyk and K. Schmudgen. *Quantum groups and their representations.* Springer, 1997.

[4] A. N. Kirillov and N. Yu. Reshetikhin. Representations of the algebra $U_q(sl(2))$, q-orthogonal polynomials and invariants of links. In *New Developments in the Theory of Knots*, pages 202–256. World Scientific, aug 1990. doi:10.1142/9789812798329_0012.

[5] Saswati Dhara, A. Mironov, A. Morozov, An Morozov, P. Ramadevi, Vivek Kumar Singh, and A. Sleptsov. Multi-Colored Links From 3-strand Braids Carrying Arbitrary Symmetric Representations. *Ann. Henri Poincare*, 20(12):4033 – 4054, 2019. arXiv:1805.03916, doi:10.1007/s00023-019-00841-z.

[6] Victor Alekseev, Andrey Morozov, and Alexey Sleptsov. Multiplicity-free $U_q(sl_N)$ 6-j symbols: Relations, asymptotics, symmetries. *Nucl. Phys. B*, 960:115164, 2020. arXiv:1912.13325, doi:10.1016/j.nuclphysb.2020.115164.

[7] Jie Gu and Hans Jockers. A note on colored HOMFLY polynomials for hyperbolic knots from WZW models. *Commun. Math. Phys.*, 338(1):393–456, 2015. arXiv:1407.5643, doi:10.1007/s00220-015-2322-z.

[8] A. Anokhina and An. Morozov. Cabling procedure for the colored HOMFLY polynomials. *Teor. Mat. Fiz.*, 178:3–68, 2014. arXiv:1307.2216, doi:10.1007/s11232-014-0129-2.

[9] William Fulton and Joe Harris. *Representation Theory. A First Course.* Springer, 1991.

[10] Vyjayanthi Chari, Andrew Pressley, et al. *A guide to quantum groups.* Cambridge university press, 1995.

[11] Philip H Butler. Coupling coefficients and tensor operators for chains of groups. *Philosophical Transactions of the Royal Society of London. Series A, Mathematical and Physical Sciences*, 277(1272):545–585, 1975.

[12] Cindy Rae Smithies. The q-deformed algebras $su(n)_q$ and their applications, 1995.

[13] James D Louck. Recent progress toward a theory of tensor operators in the unitary groups. *American Journal of Physics*, 38(1):3–42, 1970.

[14] Richard William Haase. The symmetric group and the unitary group: An application of group-subgroup transformation theory, 1983.

[15] C. Lienert and P. Butler. Racah-wigner algebra for q-deformed algebras. *Journal of Physics A*, 25:1223–1235, 1992. doi:10.1088/0305-4470/25/5/026.

[16] G. Gasper and M. Rahman. *Basic hypergeometric series.* Cambridge University Press, 1990.

[17] A. D'Adda, G. Ponzano, and R. D'Auria. Generalized 6j-coefficients and their symmetries. *Il Nuovo Cimento A (1965-1970)*, 23(1):69–102, 1974.

[18] N.Ja. Vilenkin and A.U. Klimyk. *Representation of Lie groups and special functions: Volume 3: Classical and quantum groups and special functions*, volume 3. Springer Science & Business Media, 1992.

[19] Roelof Koekoek, Peter A. Lesky, and René F. Swarttouw. *Hypergeometric orthogonal polynomials and their q-analogues*. Springer Science & Business Media, 2010.

[20] Charles F Dunkl and Yuan Xu. *Orthogonal polynomials of several variables*. Number 155. Cambridge University Press, 2014.

[21] R.A. Gustafson. A Whipple's transformation for hypergeometric series in $U(n)$ and multivariable hypergeometric orthogonal polynomials. *SIAM journal on mathematical analysis*, 18(2):495–530, 1987.

[22] N. Yu. Reshetikhin and V. G. Turaev. Ribbon graphs and their invariants derived from quantum groups. *Commun. Math. Phys.*, 127:1–26, 1990. doi:10.1007/BF02096491.

A Racah matrices from back-coupling rule

A.1 Back-coupling rule solution

In this section we find a general formula for Racah matrix of dimension 2. In this section for simplicity we assume that $q \in \mathbb{R}$, which correspond to $\star$-representations [3, 22]. In particular it allows to make Racah matrices orthogonal, as the algebra becomes real-valued. The approach is similar to [8], but we enrich it with the proof of a solution uniqueness. The statement is that the Racah back-coupling rule determines Racah matrix if the dimension equals 2.

Let us briefly recall the different forms of the Racah back-coupling rule (17). It was shown in the introduction that the Racah back-coupling rule follows from the $\mathcal{R}$-matrix definition, namely hexagon axioms (10). It can be compactly written in terms of Racah matrices as in (20). Restricting matrices only to the subspace where they act non-trivially, the equation then becomes:

$$\widetilde{\mathcal{R}}_{1,2} U_{312} \widetilde{\mathcal{R}}_{1,3} U_{231} \widetilde{\mathcal{R}}_{2,3} U_{123} = \mathbb{I} \cdot \sigma q^{\kappa_4 - \kappa_1 - \kappa_2 - \kappa_3} \tag{120}$$

This equation simplifies a lot if we normalize each $\mathcal{R}$-matrix and consider matrices of dimension two.

$$\widetilde{\mathcal{R}}_{i,j} = \begin{pmatrix} \lambda_{ij} & 0 \\ 0 & -\frac{1}{\lambda_{ij}} \end{pmatrix}, \qquad U_{123} = (u_{ij}),\ U_{231} = (v_{ij}),\ U_{312} = (w_{ij}) \tag{121}$$

The phases convention we use restrict us that $|\mathcal{R}| = -1$. Note that λ_{ij} can be either positive or negative, but the matrix always has $|\mathcal{R}| = 1$. The Racah back-coupling rule reduces to:

$$\begin{pmatrix} \lambda_{12} & 0 \\ 0 & -\frac{1}{\lambda_{12}} \end{pmatrix} \begin{pmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{pmatrix} \begin{pmatrix} \lambda_{13} & 0 \\ 0 & -\frac{1}{\lambda_{13}} \end{pmatrix} \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix} \begin{pmatrix} \lambda_{23} & 0 \\ 0 & -\frac{1}{\lambda_{23}} \end{pmatrix} \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{122}$$

This equation can be solved directly with respect to Racah matrices entries. We consider two important cases. The first one is when the representations on which act $\mathcal{R}$-matrices are equal. This case is known as the knot case. The second one is a general situation when we consider (122) in a full generality. It is called the link case.

Knot case. It implies that $\widetilde{\mathcal{R}} = \widetilde{\mathcal{R}}_{12} = \widetilde{\mathcal{R}}_{23} = \widetilde{\mathcal{R}}_{13}$ and $U = U_{123} = U_{231} = U_{312}$. So, the equation becomes:

$$(\widetilde{\mathcal{R}} U)^3 = \mathbb{I} \tag{123}$$

Note that from the determinant sign we obtain $|U|^3 = -1$, so we can write U as unitary matrix with negative determinant.

$$U = \begin{pmatrix} c & s \\ s & -c \end{pmatrix}, \qquad c^2 + s^2 = 1 \tag{124}$$

Proposition 9. *The general solutions of the knot back-coupling rule (123) is either of the form*

$$U = \begin{pmatrix} \frac{-\xi}{\xi^2+1} & \pm\frac{\sqrt{\xi^2+\xi^4+1}}{\xi^2+1} \\ \pm\frac{\sqrt{\xi^2+\xi^4+1}}{\xi^2+1} & \frac{\xi}{\xi^2+1} \end{pmatrix}, \quad \mathcal{R} = \begin{pmatrix} \lambda & 0 \\ 0 & -\frac{1}{\lambda} \end{pmatrix} \tag{125}$$

or when $\lambda = \pm 1$ it also can be

$$U = \pm \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \mathcal{R} \tag{126}$$

Proof. There are three linearly independent equations:

$$\begin{cases} \frac{s}{\lambda^3}\left(c^2(\lambda^2+1)^2 - \lambda^2\right) = 0 \\ \frac{1}{\lambda}\left(c^2\lambda^2 + c^2 - c\lambda - 1\right)\left(c(\lambda^2+1)+\lambda\right) = 0 \\ \frac{1}{\lambda^3}\left(c^2\lambda^2 + c^2 - c\lambda - \lambda^2\right)\left(c(\lambda^2+1)+\lambda\right) = 0 \end{cases} \quad \Rightarrow \quad \left[\begin{array}{l} \begin{cases} c = -\frac{\lambda}{(\lambda^2+1)}, & \lambda \in \mathbb{R} \\ s = \pm\sqrt{1-c^2} \end{cases} \\ \begin{cases} s = 0, \\ c = \lambda = \pm 1 \end{cases} \end{array}\right. \tag{127}$$

The system can be manually solved in terms of c, there are two solutions, each with some sign ambiguities. They are written on the right.

The first solution is general and can be applied for any λ. The second solution is degenerate and can occur only when $\lambda = \pm 1$. □

The signs are not fixed from the equations. This is not an issue because different signs in the solution correspond to a particular choice of basis and can be determined only by convention.

We can show that the second solution is absent if we restrict to the multiplicity-free case. We prove this fact in the more general link case.

Link case. For brevity we denote by indices a, b, c any cyclic permutation of indices $1, 2, 3$. Parameters c_a and s_a are elements of the matrix U_{cab}.

Proposition 10. *General solution of 2x2 Racah back-coupling equation* (122) *has 6 groups of solutions:*

1. $\lambda_{ab}\lambda_{bc}\lambda_{ca} = \pm 1 \quad \Rightarrow \quad c_a^2 = c_b^2 = c_c^2 = 1$
2. $\lambda_{ab}\lambda_{bc}\lambda_{ca}^{-1} = \pm 1 \quad \Rightarrow \quad s_a^2 = s_b^2 = c_c^2 = 1$
3. $\lambda_{bc}^2 = 1, \lambda_{ab}\lambda_{ca} = \pm 1 \quad \Rightarrow \quad c_a^2 = 1, s_b^2 = s_c^2$
4. $\lambda_{bc}^2 = 1, \lambda_{ab}\lambda_{ca}^{-1} = \pm 1 \quad \Rightarrow \quad c_a^2 = 1, s_b^2 = c_c^2$
5. $\lambda_{12}^2 = \lambda_{23}^2 = \lambda_{13}^2 = 1$
6. *Nontrivial solution:*

$$c = -\sqrt{\frac{(\lambda_{12}^2 - \lambda_{13}^2\lambda_{23}^2)(\lambda_{23}^2 - \lambda_{12}^2\lambda_{13}^2)}{\lambda_{13}^2(1-\lambda_{12}^4)(1-\lambda_{23}^4)}}\,, \qquad s = \sqrt{\frac{(\lambda_{13}^2 - \lambda_{12}^2\lambda_{23}^2)(1 - \lambda_{12}^2\lambda_{13}^2\lambda_{23}^2)}{\lambda_{13}^2(1-\lambda_{12}^4)(1-\lambda_{23}^4)}} \tag{128}$$

Proof. The equations can be solved with the help of computer algebra. □

A.2 Solution uniqueness

Our claim is that although there are a lot of degenerate solutions (the first five groups), all these degeneracies disappear for multiplicity-free Racah matrices. The key point is that Casimir eigenvalues of corresponding representations satisfy two properties. The first one is that $\mathcal{R}$-matrix eigenvalues satisfy inequality $\lambda_{ab} \neq \pm 1$. The second one is a triangle inequality on Casimir eigenvalues imposed by $|\lambda_{ab}\lambda_{bc}\lambda_{ca}^{\pm}| \neq \pm 1$.

Conjecture 2. *Multiplicity-free 2-dimensional Racah matrix from* $U_q(sl_N)$ *with* $q \neq 1$ *always has only a non-trivial solution of Racah back-coupling rule.*

We do not prove this statement in the paper, but a lot of examples give us a hint that it can be true. In this paper we need the more concrete version of this statement and prove it. Namely, we just show that 6-j symbols that arise as coefficients in the three-term recurrence relations satisfy only nontrivial solution of the Racah back-coupling rule.

Lemma 3. *$\mathcal{R}$-matrices acting on the pairs of representations from* $R_1 \otimes R_2 \otimes [\![1]\!]$ *has distinct eigenvalues.*

Proof. Let us consider $\mathcal{R}_{23}$ first. From here we assume that $q \neq 1$ and the product of representations is multiplicity-free. The $\mathcal{R}$-matrices then have eigenvalues:

$$\lambda_{ab} = q^{\frac{1}{2}(\kappa_{Q_1} - \kappa_{Q_2})} := q^{\kappa_{ab}} \qquad \text{for } Q_i \subset R_a \otimes R_b,\ Q \subset Q_i \otimes R_c, \qquad \kappa_\mu = \sum_{(i,j)\in\mu} (i-j) \tag{129}$$

Here κ are the second Casimir eigenvalues. It acts on the space $R_2 \otimes [1]$, which can be decomposed on irreducible representations in a multiplicity-free way. Two eigenvalues correspond to diagrams, say, μ_1 and μ_2. These diagrams are almost the same partitions: they differ by the displacement of the only element. From multiplicity-free condition we can state that the displacement is non-zero. Thus, the sum in κ_{23} reduces to the only non-zero term:

$$2\kappa_{23} = \sum_{(i,j)\in\mu_1} (i-j) - \sum_{(i,j)\in\mu_2} (i-j) = (i_1 - i_2 - j_1 + j_2) \geq 2 \tag{130}$$

The Young diagrams μ_1 and μ_2 are lexicographically ordered, so $i_1 > i_2$ and $j_1 > j_2$. Thus, the eigenvalues of $\mathcal{R}_{23}$ are not ± 1.

The eigenvalues of $\mathcal{R}_{12}$ are obtained in the following way. Let us again denote the Young diagrams corresponding to eigenvalues by μ_1 and μ_2. Then we can connect them because $\mu_1 \otimes [\![1]\!] \supset Q \subset \mu_2 \otimes [\![1]\!]$. It means that both μ_1 and μ_2 differ from Q by one element and hence κ_{12} reduces to a single term and again it is always positive:

$$\kappa_{12} = \frac{1}{2}(i_1 - i_2 - j_1 + j_2) \geq 1 \tag{131}$$

The derivation for $\mathcal{R}_{13}$ is identical. □

This lemma shows that the solutions from groups $3, 4, 5$ are absent for considered representations. To handle the first two groups we need the following lemma.

Lemma 4. *For the 2-dimensional Racah matrix of the form:*

$$U \begin{pmatrix} [\![a, b^{N-2}]\!] & [\![s]\!] \\ [\![1]\!] & [\![c, d^{N-2}]\!] \end{pmatrix} \tag{132}$$

the solution is unique and non-trivial. In particular, there is no degeneracy $\pm\Delta\kappa_{12} \pm \Delta\kappa_{23} \pm \Delta\kappa_{13} = 0$.

Proof. This matrix is of dimension at most 2, it can be deduced from fusion rules. The corresponding set of three $\mathcal{R}$-matrices has the same dimension. It is possible to write explicitly the representations on which each matrix acts.

The representations $Q_i^{(12)}$ for $\mathcal{R}_{12}$ are found from $[\![a, b^{N-2}]\!] \otimes [\![s]\!] \supset Q_i^{(12)} \subset [\![c, d^{N-2}]\!] \otimes \overline{[\![1]\!]}$. Using Littlewood-Richardson rules we derive the diagrams and obtain the eigenvalues:

$$Q_1^{(12)} = [\![c+1, (d+1)^{N-2}]\!], \qquad Q_2^{(12)} = [\![c, (d+1)^{N-2}, 1]\!], \qquad \lambda_{12} = q^{\frac{1}{2}(c+N-1)} \tag{133}$$

The same procedure can be done for $\mathcal{R}_{23}$ and $\mathcal{R}_{13}$:

$$\begin{array}{ll} Q_1^{(23)} = [\![s+1]\!], \qquad Q_2^{(23)} = [\![s,1]\!], \qquad \lambda_{23} = q^{\frac{1}{2}(s+1)} \\ Q_1^{(13)} = [\![a+1, b^{N-2}]\!], \qquad Q_2^{(13)} = [\![a, b^{N-2}, 1]\!], \qquad \lambda_{13} = q^{\frac{1}{2}(a+N-1)} \end{array} \tag{134}$$

We can see that each eigenvalue is non-degenerate in a sense that $\kappa_{ab} \neq 0$. To go further we need to note that the Racah matrix is non-trivial only if additional fusion rules are imposed on c. For a fixed a, b, s we can obtain that $[\![c, d^{N-2}]\!]$ is of the form:

$$[\![a, b^{N-2}]\!] \otimes [\![s]\!] \otimes [\![1]\!] = \bigoplus_{i=0}^{\min(s+1,b)} [\![a+s+1-i, b^{N-2}, i]\!] \tag{135}$$

From this decomposition we obtain that $\max(0, a-s-1) \leq c \leq a+s+1$. As $\kappa_{ab} > 0$, we need to check only thee cases of $\pm\kappa_{12} \pm \kappa_{23} \pm \kappa_{13} \neq 0$:

$$\begin{aligned} -\kappa_{12} + \kappa_{23} + \kappa_{13} &= a+s+1-(a+s+1-2i) = 2i \geq 0 \\ \kappa_{12} - \kappa_{23} + \kappa_{13} &= a-s-1+2(N-1)+a+s+1-2i = 2(N-1)+2a-2i \geq 2(N-1+a-b) \geq 1 \\ \kappa_{12} + \kappa_{23} - \kappa_{13} &= s+1-a+(a+s+1-2i) = 2(s+1+i) \geq 0 \end{aligned} \tag{136}$$

The equality $\pm\kappa_{12} \pm \kappa_{23} \pm \kappa_{13} = 0$ is satisfied only for maximal and minimal i, but these values make Racah matrix one-dimensional. Hence, for two-dimensional Racah matrix of considered type there are no degeneracy and the solution is unique and non-trivial. As the product of diagrams is commutative and associative, the same result is true for arbitrary permutation of the arguments $[\![a, b^{N-2}]\!], [\![s]\!], [\![1]\!]$ in Racah matrix. $\square$

Lemma 5. *For the following 2-dimensional Racah matrix the solution is unique and non-trivial:*

$$U \begin{pmatrix} [\![a,b]\!] & \overline{[\![s]\!]} \\ [\![1]\!] & [\![c, d^{N-2}]\!] \end{pmatrix} \tag{137}$$

Proof. The proof is analogous to the previous lemma. $\square$

A.3 Two-dimensional Racah matrices

Let us consider the the non-trivial solution of (122) and write it in a form appropriate for our computations. The Racah matrix is:

$$U_{123} = \begin{pmatrix} -\sqrt{\dfrac{(\lambda_{12}^2 - \lambda_{13}^2\lambda_{23}^2)(\lambda_{23}^2 - \lambda_{12}^2\lambda_{13}^2)}{\lambda_{13}^2(1-\lambda_{12}^4)(1-\lambda_{23}^4)}} & \sqrt{\dfrac{(\lambda_{13}^2 - \lambda_{12}^2\lambda_{23}^2)(1 - \lambda_{12}^2\lambda_{13}^2\lambda_{23}^2)}{\lambda_{13}^2(1-\lambda_{12}^4)(1-\lambda_{23}^4)}} \\ \sqrt{\dfrac{(\lambda_{13}^2 - \lambda_{12}^2\lambda_{23}^2)(1 - \lambda_{12}^2\lambda_{13}^2\lambda_{23}^2)}{\lambda_{13}^2(1-\lambda_{12}^4)(1-\lambda_{23}^4)}} & \sqrt{\dfrac{(\lambda_{12}^2 - \lambda_{13}^2\lambda_{23}^2)(\lambda_{23}^2 - \lambda_{12}^2\lambda_{13}^2)}{\lambda_{13}^2(1-\lambda_{12}^4)(1-\lambda_{23}^4)}} \end{pmatrix} \tag{138}$$

where λ are normalized $\mathcal{R}$-matrix eigenvalues and they are chosen to be always greater or equal to one.

$$\lambda_i = q^{\kappa_{Q_i} - \kappa_{R_1} - \kappa_{R_2}} \quad \text{for } Q_i \subset R_1 \otimes R_2, Q \subset Q_i \otimes R_1, \qquad \lambda = q^{\frac{1}{2}(\kappa_{Q_1} - \kappa_{Q_2})} = q^{\kappa_{12}} \tag{139}$$

The matrix entries can be written as a q-numbers:

$$\lambda_{12}^2 - \lambda_{13}^2\lambda_{23}^2 = \lambda_{12}\lambda_{13}\lambda_{23}\left(\frac{\lambda_{12}}{\lambda_{13}\lambda_{23}} - \frac{\lambda_{13}\lambda_{23}}{\lambda_{12}}\right) = \lambda_{12}\lambda_{13}\lambda_{23}\left(q - \frac{1}{q}\right)[\kappa_{12} - \kappa_{13} - \kappa_{23}] \tag{140}$$

So the matrix becomes:

$$U_{123} = \begin{pmatrix} -\sqrt{\dfrac{[\kappa_{12}-\kappa_{13}-\kappa_{23}]\,[\kappa_{23}-\kappa_{12}-\kappa_{13}]}{[2\kappa_{12}]\,[2\kappa_{23}]}} & \sqrt{\dfrac{[\kappa_{12}+\kappa_{23}-\kappa_{13}]\,[\kappa_{12}+\kappa_{23}+\kappa_{13}]}{[2\kappa_{12}]\,[2\kappa_{23}]}} \\ \sqrt{\dfrac{[\kappa_{12}+\kappa_{23}-\kappa_{13}]\,[\kappa_{12}+\kappa_{23}+\kappa_{13}]}{[2\kappa_{12}]\,[2\kappa_{23}]}} & \sqrt{\dfrac{[\kappa_{12}-\kappa_{13}-\kappa_{23}]\,[\kappa_{23}-\kappa_{12}-\kappa_{13}]}{[2\kappa_{12}]\,[2\kappa_{23}]}} \end{pmatrix} \tag{141}$$

Example 5. Let us compute the symbol:

$$\begin{Bmatrix} [\![a, b^{N-2}]\!] & [\![s_2]\!] & [\![s_{123}-1]\!] \\ [\![1]\!] & [\![s_{123}]\!] & [\![s_2+1]\!] \end{Bmatrix} \tag{142}$$

The first we need to do is to obtain Young diagrams corresponding to $\mathcal{R}_{12}, \mathcal{R}_{23}$ and $\mathcal{R}_{13}$:

$$\begin{aligned} &[\![s_{123}]\!] \otimes \overline{[\![1]\!]} = [\![s_{123}+1, 1^{N-2}]\!] \oplus [\![s_{123}, 1^{N-1}]\!] && 2\kappa_{12} = s_{123} + N - 1, \\ &[\![s_2]\!] \otimes [\![1]\!] = [\![s_2+1]\!] \oplus [\![s_2, 1]\!] && 2\kappa_{23} = s_2 + 1, \\ &[\![a, b^{N-2}]\!] \otimes [\![1]\!] = [\![a+1, b^{N-2}]\!] \oplus [\![a, b^{N-2}, 1]\!] && 2\kappa_{13} = a + N - 1 \end{aligned} \tag{143}$$

We recall that the basis in for Racah matrix is constructed from lexicographically ordered Young matrices. Thus, we obtain the following value:

$$\left|\begin{Bmatrix} [\![a, b^{N-2}]\!] & [\![s_2]\!] & [\![s_{123}-1]\!] \\ [\![1]\!] & [\![s_{123}]\!] & [\![s_2+1]\!] \end{Bmatrix}\right|^2 = \frac{[\frac{s_2+s_{123}-a}{2}][\frac{a+s_{123}+s_2-1}{2} + N]}{\dim([\![s_{123}-1]\!])\dim([\![s_2+1]\!])[s_{123}+N-1][s_2+1]} \tag{144}$$

We finish this example with the simplifying result:

For Racah matrices computation we use the following simple expression:

$$|U_k|^2 = \frac{[\kappa_{23} + k(\kappa_{13} - \kappa_{12})][\kappa_{12} + \kappa_{13} - k(\kappa_{23})]}{[2\kappa_{12}][2\kappa_{23}]} \qquad k : \begin{pmatrix} + & - \\ - & + \end{pmatrix} \tag{145}$$

The sign k is chosen for $(1,1)$ and $(2,2)$ elements of matrix as $+1$ and -1 for the other two. The eigenvalues are calculated above and there are basically three cases:

$$2\kappa_{12} = s \qquad R_1 \otimes R_2 = [s-1, 1] \oplus [s] \tag{146}$$

$$2\kappa_{12} = s + N \qquad R_1 \otimes R_2 = [s-1, 1^{N-2}] \oplus [s] \tag{147}$$

$$2\kappa_{12} = r + N \qquad R_1 \otimes R_2 = [r, c^{N-2}] \oplus [r+2, (c+1)^{n-2}] \tag{148}$$